Kinematical Lie algebras and contact sub-Riemannian symmetric spaces

Pierre Bieliavsky (joint work with Nicolas Boulanger)

Prague, 22/04/2026

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Current joint project with A.A. : "symplectization" of Mackey's bijection

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Historical introduction I: relativity principles

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Newton's first law

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[Figueroa-O'Farril 2018]: classification of $\mathfrak{g}(D)$'s.

Historical introduction II: Carnot-Caratheodory spaces

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Caratheodory [1909] formulates the First Law of Thermodynamics in terms of "horizontality" of adiabatic processes.

Definition. (M. Gromov [1981]) A **C-C space** is a smooth manifold M equipped with a tangent distribution $\mathcal{D} \subset T(M)$.

Theorem. [Rashevskii (1938) - Chow (1939), Gromov (1989)] When $\mathcal{D}$ generates (under bracket) the whole $T(M)$, every point x possesses a neighbourhood where every point can be joined to x by a horizontal curve.

Example (Heisenberg group)

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$\mathcal{D}$ is spanned by $e := \partial_y$ and $f := y\partial_x - \partial_z$.

e , f and $[e, f] = \partial_x$ span a Heisenberg Lie algebra

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"Symmetric" C-C spaces are (generalized) Kinematical Lie algebras.

Symmetric spaces [O. Loos, 1969]

Definition. A **symmetric space** is a pair (M, s) where M is a smooth manifold and $s : M \times M \to M : (x, y) \mapsto s_x(y)$ is a smooth map, such that

1. $s_x^2 = \mathtt{Id}_M$.
2. x is an isolated fixed point of s_x.
3. $s_x\, s_y\, s_x \;=\; s_{s_x(y)}$.

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Proposition. [Loos] There exists a unique linear connection ∇ in $T(M)$ that is invariant under the symmetries.
Moreover:

$$T^\nabla \;\equiv\; 0 \quad \text{and} \quad \nabla R^\nabla \;\equiv\; 0\,.$$

Proposition. [B. (2000)] The expression

$$(\nabla_X Y)_x \;:=\; \frac{1}{2}\,[X\,,\; Y \,+\, s_{x\star} Y]_x$$

defines the Loos connection in $T(M)$.

Symplectic symmetric spaces [B.-Cahen-Gutt, 1994]

A **symplectic symmetric space** is a triple (M, s, ω) where (M, s) is a symmetric space and ω is a **non-degenerate differential 2-form** on M which is invariant under the symmetries:

$$s_x^\star \omega \;=\; \omega\ . \tag{1}$$

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Example: Massive (coadjoint) orbits of the Poincaré group are SSS. Their symplectic connection ∇ is *never* metric (i.e. Levi-Civita).

Definition. A differential one-form θ on a connected smooth manifold $\mathcal{M}$ is a **contact form** if

$$\mathrm{d}\,\theta\,|_{\mathcal{D}\times\mathcal{D}} \quad \text{is non-degenerate on} \quad \mathcal{D} \;:=\; \ker(\theta)\;.$$

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When equipped with a smoothly varying **Riemannian partial metric** on $\mathcal{D}$

$$g \in \underline{\mathcal{D}^\star \otimes \mathcal{D}^\star}\ ,$$

the triple $(\mathcal{M}, \mathcal{D}, g)$ is called a **contact sub-Riemannian** manifold.

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A **sub-isometry** is a diffeomorphism of $\mathcal{M}$ preserving the distribution and the partial metric.

The adapted connection [Falbel-Gorodski 1994]

Theorem: Let $(\mathcal{M}, \mathcal{D}, g)$ be a contact sub-Riemannian manifold. Then there exists a unique linear connection ∇ in $T(\mathcal{M})$ such that

1. $\nabla : \underline{T(M)} \times \underline{\mathcal{D}} \to \underline{\mathcal{D}}$
2. $\nabla\xi = 0$
3. $\nabla g = 0$
4. $T^{\nabla}(X, Y) = \mathrm{d}\,\theta(X, Y)\,\xi$

The element $\tau \in \underline{\mathtt{End}(\mathcal{D})}$ defined by

$$\tau(X) := T^{\nabla}(\xi, X)$$

is called the **(sub-torsion)**.

Proposition [B.-Falbel-Gorodski-Tausk, 1995] Every two points in $\mathcal{M}$ can be joined by a horizontal broken ∇-geodesic.

sub-Riemannian contact symmetric spaces [Strichartz 1986]

A **sub-Riemannian contact symmetric space** is a quadruple $(\mathcal{M}, \mathcal{D}, g, \psi)$ where

$$\psi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

is a (volume preserving) smooth map such that, for every $x \in M$, the map

$$\psi_x : \mathcal{M} \to \mathcal{M} : y \mapsto \psi(x, y)$$

is a **sub-isometry** that fixes x and such that

$$\psi_{x_{\star x}}|_{\mathcal{D}_x} \;=\; -\mathtt{Id}_{\mathcal{D}_x} \;.$$

Classification of sub-Riemannian contact symmetric spaces [B.-Falbel-Gorodski, Pacific Jour. Math. (1995)]

Theorem: The Reeb field of a sub-Riemannian contact symmetric space $(\mathcal{M}, \mathcal{D}, g, \psi)$ induces a principal fibration of $\mathcal{M}$ over a symplectic symmetric space M ($\dim(\mathcal{M}) = \dim(M) + 1$).

type			*examples*	*sub-torsion*	*holonomy*
solvable			H^{2n+1}	zero	trivial
semisimple	Hermitean		S^1-fibration over Hermitean Riemannian symmetric space	zero	irreducible if symmetric space is irreducible
	non-Hermitean	simple	$SO(n+2)/SO(n)$ $(n \geq 3)$	nonzero	irreducible
			$SO(n,2)/SO(n)$ $(n \geq 3)$	nonzero	irreducible
			$SO(n+1,1)/SO(n)$	nonzero	irreducible if $n \geq 3$
		non-simple	$SO(4)/SO(2)$	nonzero	not irreducible
			$SO(2,2)/SO(2)$	nonzero	not irreducible
else			$SO(n+1) \bowtie R^{n+1}/SO(n)$	nonzero	irreducible if $n \geq 3$
			$SO(n,1) \bowtie R^{n+1}/SO(n)$	nonzero	irreducible if $n \geq 3$
			twisted product of H^{2n+1} and Hermitean	zero	not irreducible

Table 1. Contact sub-Riemannian symmetric spaces of dimension $2n+1 \geq 5$

Main result [B.-Boulanger, Lett. Math. Phys. (2025)]

Definition. A *generic kinematical Lie algebra* is a finite dimensional real Lie algebra $\mathfrak{g}$ that admits a vector space decomposition:

$$\mathfrak{g} \;=\; \mathcal{K} \,\oplus \mathcal{Z} \oplus (V \oplus V)$$

such that

1. $\mathcal{K}$ is a (compact) Lie sub-algebra.
2. V is a simple faithful $\mathcal{K}$-module.
3. The isotypical component of V in $\Lambda^2(V)$ is empty ($D \geq 4$).
4. $\mathcal{Z} \simeq \mathbb{R}$ is one-dimensional.
5. $[\mathcal{Z}\,, \mathcal{K}] \;=\; \{0\}$.

Theorem: Let G be a (connected simply connected) generic kinematical Lie group. Then:

1. $\mathcal{M} := G/K$ is a sub-Riemannian contact symmetric space, while $M := G/H$ is a symplectic symmetric space.
2. The principal fibration $G/K \to G/H : gK \mapsto gH$ realizes the Reeb fibration of $\mathcal{M}$ over its associated symplectic symmetric space.
3. A sub-Riemannian contact symmetric space comes from such a generalized Kinematical Lie group that way if and only if its horizontal holonomy acts reducibly.