Homological reduction of Poisson structures

Pedro H. Carvalho

University of Hradec Králové

Context

Consider a Poisson manifold (M, π) endowed with a hamiltonian $\mathfrak{g}$-action with moment map $J : M \to \mathfrak{g}^*$.

Context

Consider a Poisson manifold (M, π) endowed with a hamiltonian $\mathfrak{g}$-action with moment map $J : M \to \mathfrak{g}^*$. Then

BFV in degree zero

Kostant-Sternberg BRST algebra: `dg super Poisson algebra` whose cohomology in degree zero recovers the Poisson algebra of the reduced space (C_{red}, π_{red}) associated to the level set $C := J^{-1}(0)$, for $0 \in \mathfrak{g}^*$ a regular value.

Context

Consider a Poisson manifold (M, π) endowed with a hamiltonian $\mathfrak{g}$-action with moment map $J : M \to \mathfrak{g}^*$. Then

BFV in degree zero

Kostant-Sternberg BRST algebra: `dg super Poisson algebra` whose cohomology in degree zero recovers the Poisson algebra of the reduced space (C_{red}, π_{red}) associated to the level set $C := J^{-1}(0)$, for $0 \in \mathfrak{g}^*$ a regular value.

Idea

To study BFV models for the hamiltonian reduction of graded symplectic manifolds of degree one with a view towards `homological reduction of Poisson structures`.

Poisson manifolds (M, π)
Symplectic $\mathbb{N}Q$-manifolds of degree one $\big(\mathcal{M} := T^*[1]M, \{\cdot,\cdot\}, X_\pi = \{\pi, \cdot\}\big)$

Poisson manifolds (M, π)
Symplectic $\mathbb{N}Q$-manifolds of degree one $\big(\mathcal{M} := T^*[1]M, \{\cdot,\cdot\}, X_\pi = \{\pi, \cdot\}\big)$

Reduction of (M, π)
Coisotropic and presymplectic reduction of $\big(\mathcal{M} := T^*[1]M, \{\cdot,\cdot\}, X_\pi = \{\pi, \cdot\}\big)$

A generalized hamiltonian setting

A generalized hamiltonian setting

- Following Cattaneo-Zambon graded geometric approach to Poisson reduction:

A generalized hamiltonian setting

- Following Cattaneo-Zambon graded geometric approach to Poisson reduction:

hamiltonian action of $\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}$ on $T^*[1]M$ *compatible* with $\pi \in C_2^\infty(T^*[1]M)$
$\mathfrak{g}$-action on (M, π) with $J : M \to \mathfrak{h}^*$ equivariant for $\mathfrak{h}$ a $\mathfrak{g}$-module

- Following Cattaneo-Zambon graded geometric approach to Poisson reduction:

hamiltonian action of $\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}$ on $T^*[1]M$ *compatible* with $\pi \in C^\infty_2(T^*[1]M)$
$\mathfrak{g}$-action on (M, π) with $J : M \to \mathfrak{h}^*$ equivariant for $\mathfrak{h}$ a $\mathfrak{g}$-module

Theorem (Cattaneo-Zambon)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the pair (ψ, J) is regular and compatible with the Poisson structure π. Then the quotient $C_{red} := C/G$ inherits a Poisson structure π_{red}.

A generalized hamiltonian setting

- Following Cattaneo-Zambon graded geometric approach to Poisson reduction:

hamiltonian action of $\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}$ on $T^*[1]M$ *compatible* with $\pi \in C_2^\infty(T^*[1]M)$
$\mathfrak{g}$-action on (M, π) with $J : M \to \mathfrak{h}^*$ equivariant for $\mathfrak{h}$ a $\mathfrak{g}$-module

Theorem (Cattaneo-Zambon)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the pair (ψ, J) is regular and compatible with the Poisson structure π. Then the quotient $C_{red} := C/G$ inherits a Poisson structure π_{red}.

`What is the corresponding homological version of this result?`

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

- Marsden-Weinstein theorem: $(J^{-1}(0)/G, \pi_{red})$;

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

- Marsden-Weinstein theorem: $(J^{-1}(0)/G, \pi_{red})$;
- Let $C := J^{-1}(0)$ and $C_{red} = C/G$.

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

- Marsden-Weinstein theorem: $(J^{-1}(0)/G, \pi_{red})$;
- Let $C := J^{-1}(0)$ and $C_{red} = C/G$. We have

$$C^\infty(C_{red}) \cong C^\infty(C)^G.$$

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

- Marsden-Weinstein theorem: $(J^{-1}(0)/G, \pi_{red})$;
- Let $C := J^{-1}(0)$ and $C_{red} = C/G$. We have

$$C^\infty(C_{red}) \cong C^\infty(C)^G.$$

- $I \subset C^\infty(M)$, the vanishing ideal of C, satisfies $\{I, I\} \subset I$.

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

- Marsden-Weinstein theorem: $(J^{-1}(0)/G, \pi_{red})$;
- Let $C := J^{-1}(0)$ and $C_{red} = C/G$. We have

$$C^\infty(C_{red}) \cong C^\infty(C)^G.$$

- $I \subset C^\infty(M)$, the vanishing ideal of C, satisfies $\{I, I\} \subset I$. Moreover,

$$C^\infty(C) \cong C^\infty(M)/I \tag{1}$$

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

- Marsden-Weinstein theorem: $(J^{-1}(0)/G, \pi_{red})$;
- Let $C := J^{-1}(0)$ and $C_{red} = C/G$. We have

$$C^\infty(C_{red}) \cong C^\infty(C)^G.$$

- $I \subset C^\infty(M)$, the vanishing ideal of C, satisfies $\{I, I\} \subset I$. Moreover,

$$C^\infty(C) \cong C^\infty(M)/I \tag{1}$$

and

$$(C^\infty(M)/I)^G \cong N(I)/I, \tag{2}$$

where

$$N(I) = \{f \in C^\infty(M) : \{f, I\} \subset I\}.$$

Let (M, π) be a Poisson manifold endowed with a $\mathfrak{g}$-action, and let $J : M \to \mathfrak{g}^*$ be a moment map.

- Marsden-Weinstein theorem: $(J^{-1}(0)/G, \pi_{red})$;
- Let $C := J^{-1}(0)$ and $C_{red} = C/G$. We have

$$C^\infty(C_{red}) \cong C^\infty(C)^G.$$

- $I \subset C^\infty(M)$, the vanishing ideal of C, satisfies $\{I, I\} \subset I$. Moreover,

$$C^\infty(C) \cong C^\infty(M)/I \tag{1}$$

  and

$$(C^\infty(M)/I)^G \cong N(I)/I, \tag{2}$$

  where

$$N(I) = \{f \in C^\infty(M) : \{f, I\} \subset I\}.$$

A *homological model* for hamiltonian reduction comes out of realizing (1) and (2) homologically.

The Kostant-Sternberg BRST algebra is a differential graded Poisson algebra $(\mathcal{K}^\bullet, \{\cdot,\cdot\}, \partial_{BRST})$, with $\mathcal{K}^\bullet := \bigoplus_{n\in\mathbb{Z}} \mathcal{K}^n$, where

$$\mathcal{K}^n := \bigoplus_{n=p-q} K^{p,q}, \quad \text{for} \quad K^{p,q} := C^\infty(M) \otimes \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^q \mathfrak{g},$$

The Kostant-Sternberg BRST algebra is a `differential graded Poisson algebra` $(\mathcal{K}^\bullet, \{\cdot,\cdot\}, \partial_{BRST})$, with $\mathcal{K}^\bullet := \bigoplus_{n\in\mathbb{Z}} \mathcal{K}^n$, where

$$\mathcal{K}^n := \bigoplus_{n=p-q} K^{p,q}, \quad \text{for} \quad K^{p,q} := C^\infty(M) \otimes \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^q \mathfrak{g},$$

and $\partial_{BRST} = \{Q_\mathfrak{g}, \cdot\}$, for

$$Q_\mathfrak{g} = \mu^*(u^i)u_i^* - \frac{1}{2} c_k^{ij} u_i^* u_j^* u^k, \quad Q_\mathfrak{g} \in \mathcal{K}^1, \quad \{Q_\mathfrak{g}, Q_\mathfrak{g}\} = 0.$$

The Kostant-Sternberg BRST algebra is a differential graded Poisson algebra $(\mathcal{K}^\bullet, \{\cdot,\cdot\}, \partial_{BRST})$, with $\mathcal{K}^\bullet := \bigoplus_{n\in\mathbb{Z}} \mathcal{K}^n$, where

$$\mathcal{K}^n := \bigoplus_{n=p-q} K^{p,q}, \quad \text{for} \quad K^{p,q} := C^\infty(M) \otimes \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^q \mathfrak{g},$$

and $\partial_{BRST} = \{Q_\mathfrak{g}, \cdot\}$, for

$$Q_\mathfrak{g} = \mu^*(u^i)u_i^* - \frac{1}{2} c_k^{ij} u_i^* u_j^* u^k, \quad Q_\mathfrak{g} \in \mathcal{K}^1, \quad \{Q_\mathfrak{g}, Q_\mathfrak{g}\} = 0.$$

Theorem (Kostant-Sternberg, Stasheff)

Let $I \subset C^\infty(M)$ be the vanishing ideal of $C := \mu^{-1}(0)$.

The Kostant-Sternberg BRST algebra is a `differential graded Poisson algebra` $(\mathcal{K}^\bullet, \{\cdot,\cdot\}, \partial_{BRST})$, with $\mathcal{K}^\bullet := \bigoplus_{n\in\mathbb{Z}} \mathcal{K}^n$, where

$$\mathcal{K}^n := \bigoplus_{n=p-q} K^{p,q}, \quad \text{for} \quad K^{p,q} := C^\infty(M) \otimes \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^q \mathfrak{g},$$

and $\partial_{BRST} = \{Q_\mathfrak{g}, \cdot\}$, for

$$Q_\mathfrak{g} = \mu^*(u^i)u_i^* - \frac{1}{2} c_k^{ij}\, u_i^* u_j^* u^k, \quad Q_\mathfrak{g} \in \mathcal{K}^1, \quad \{Q_\mathfrak{g}, Q_\mathfrak{g}\} = 0.$$

Theorem (Kostant-Sternberg, Stasheff)

Let $I \subset C^\infty(M)$ be the vanishing ideal of $C := \mu^{-1}(0)$. Then

$$H^0_{\{Q_\mathfrak{g},\cdot\}} \cong N(I)/I$$

The Kostant-Sternberg BRST algebra is a `differential graded Poisson algebra` $(\mathcal{K}^\bullet, \{\cdot,\cdot\}, \partial_{BRST})$, with $\mathcal{K}^\bullet := \bigoplus_{n\in\mathbb{Z}} \mathcal{K}^n$, where

$$\mathcal{K}^n := \bigoplus_{n=p-q} K^{p,q}, \quad \text{for} \quad K^{p,q} := C^\infty(M) \otimes \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^q \mathfrak{g},$$

and $\partial_{BRST} = \{Q_\mathfrak{g}, \cdot\}$, for

$$Q_\mathfrak{g} = \mu^*(u^i)u_i^* - \frac{1}{2} c_k^{ij}\, u_i^* u_j^* u^k, \quad Q_\mathfrak{g} \in \mathcal{K}^1, \quad \{Q_\mathfrak{g}, Q_\mathfrak{g}\} = 0.$$

Theorem (Kostant-Sternberg, Stasheff)

Let $I \subset C^\infty(M)$ be the vanishing ideal of $C := \mu^{-1}(0)$. Then

$$H^0_{\{Q_\mathfrak{g},\,\cdot\,\}} \cong N(I)/I \cong C^\infty(C_{red}).$$

A generalized hamiltonian setting

- Following Cattaneo-Zambon graded geometric approach to Poisson reduction:

hamiltonian action of $\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}$ on $T^*[1]M$ *compatible* with $\pi \in C_2^\infty(T^*[1]M)$
$\mathfrak{g}$-action on (M, π) with $J : M \to \mathfrak{h}^*$ equivariant, for $\mathfrak{h}$ a $\mathfrak{g}$-module

Theorem (Cattaneo-Zambon)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the pair (ψ, J) is regular and compatible with the Poisson structure π. Then the quotient $C_{red} := C/G$ inherits a Poisson structure π_{red}.

A generalized hamiltonian setting

- Following Cattaneo-Zambon graded geometric approach to Poisson reduction:

hamiltonian action of $\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}$ on $T^*[1]M$ *compatible* with $\pi \in C_2^\infty(T^*[1]M)$
$\mathfrak{g}$-action on (M, π) with $J : M \to \mathfrak{h}^*$ equivariant, for $\mathfrak{h}$ a $\mathfrak{g}$-module

Theorem (Cattaneo-Zambon)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the pair (ψ, J) is regular and compatible with the Poisson structure π. Then the quotient $C_{red} := C/G$ inherits a Poisson structure π_{red}.

`What is the algebraic structure of the homological model for this reduction scheme?`

Shifted cotangent bundles

Shifted cotangent bundles

For M a smooth manifold, the *shifted cotangent bundle* of M is the degree one manifold $\mathcal{M} := T^*[1]M$

Shifted cotangent bundles

For M a smooth manifold, the *shifted cotangent bundle* of M is the degree one manifold $\mathcal{M} := T^*[1]M$ whose sheaf of functions is

$$C^\infty(\mathcal{M}) := \Gamma(\bigwedge^{\bullet} TM) = \mathfrak{X}^\bullet(M).$$

Shifted cotangent bundles

For M a smooth manifold, the *shifted cotangent bundle* of M is the degree one manifold $\mathcal{M} := T^*[1]M$ whose sheaf of functions is

$$C^\infty(\mathcal{M}) := \Gamma(\bigwedge^{\bullet} TM) = \mathfrak{X}^\bullet(M).$$

In local coordinates $(U, x_1, \dots, x_n)$ for M, a multivector field $X \in \mathfrak{X}^p(M)$ can be written as

$$X = \sum_{i_1 < \cdots < i_p} a_{i_1 \dots i_p} \xi_{i_1} \dots \xi_{i_p}, \quad a_{i_1 \dots i_p} \in C^\infty(M)|_U,$$

where $\xi_{i_j} := \partial_{x_{i_j}}$.

Shifted cotangent bundles

For M a smooth manifold, the *shifted cotangent bundle* of M is the degree one manifold $\mathcal{M} := T^*[1]M$ whose sheaf of functions is

$$C^\infty(\mathcal{M}) := \Gamma(\bigwedge^\bullet TM) = \mathfrak{X}^\bullet(M).$$

In local coordinates $(U, x_1, \ldots, x_n)$ for M, a multivector field $X \in \mathfrak{X}^p(M)$ can be written as

$$X = \sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p} \xi_{i_1} \ldots \xi_{i_p}, \quad a_{i_1 \ldots i_p} \in C^\infty(M)|_U,$$

where $\xi_{i_j} := \partial_{x_{i_j}}$.

- $X \in \mathfrak{X}^p(M)$ and $Y \in \mathfrak{X}^q(M)$, the shifted Poisson bracket $\{X, Y\} \in \mathfrak{X}^{p+q-1}(M)$

Shifted cotangent bundles

For M a smooth manifold, the *shifted cotangent bundle* of M is the degree one manifold $\mathcal{M} := T^*[1]M$ whose sheaf of functions is

$$C^\infty(\mathcal{M}) := \Gamma(\bigwedge^\bullet TM) = \mathfrak{X}^\bullet(M).$$

In local coordinates $(U, x_1, \dots, x_n)$ for M, a multivector field $X \in \mathfrak{X}^p(M)$ can be written as

$$X = \sum_{i_1 < \cdots < i_p} a_{i_1 \dots i_p} \xi_{i_1} \dots \xi_{i_p}, \quad a_{i_1 \dots i_p} \in C^\infty(M)|_U,$$

where $\xi_{i_j} := \partial_{x_{i_j}}$.

- $X \in \mathfrak{X}^p(M)$ and $Y \in \mathfrak{X}^q(M)$, the shifted Poisson bracket $\{X, Y\} \in \mathfrak{X}^{p+q-1}(M)$ is given, in local coordinates, by

$$\{X, Y\} = \sum_i \frac{\partial X}{\partial \xi_i} \frac{\partial Y}{\partial x_i} - (-1)^{(p-1)(q-1)} \sum_i \frac{\partial Y}{\partial \xi_i} \frac{\partial X}{\partial x_i}.$$

Hamiltonian reduction of $\mathcal{M} := T^*[1]M$

Hamiltonian reduction of $\mathcal{M} := T^*[1]M$

- Let $\overline{\mathfrak{g}} = \mathfrak{h}[1] \oplus \mathfrak{g}$ be a graded Lie algebra concentrated in degrees -1 and 0. An action of $\overline{\mathfrak{g}}$ on $\mathcal{M}$ is a morphism of graded Lie algebras $\Psi : \overline{\mathfrak{g}} \to \mathfrak{X}(\mathcal{M})$.

Hamiltonian reduction of $\mathcal{M} := T^*[1]M$

- Let $\overline{\mathfrak{g}} = \mathfrak{h}[1] \oplus \mathfrak{g}$ be a graded Lie algebra concentrated in degrees -1 and 0. An action of $\overline{\mathfrak{g}}$ on $\mathcal{M}$ is a morphism of graded Lie algebras $\Psi : \overline{\mathfrak{g}} \to \mathfrak{X}(\mathcal{M})$.
- Moment map: a morphism of (odd) Lie algebras $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ such that

$$u_\mathcal{M} = \{J_1^\sharp(u), \cdot\} \text{ and } v_\mathcal{M} = \{J_0^\sharp(v), \cdot\},$$

for $u \in \mathfrak{g}$ and $v \in \mathfrak{h}$.

Hamiltonian reduction of $\mathcal{M} := T^*[1]M$

- Let $\overline{\mathfrak{g}} = \mathfrak{h}[1] \oplus \mathfrak{g}$ be a graded Lie algebra concentrated in degrees -1 and 0. An action of $\overline{\mathfrak{g}}$ on $\mathcal{M}$ is a morphism of graded Lie algebras $\Psi : \overline{\mathfrak{g}} \to \mathfrak{X}(\mathcal{M})$.
- `Moment map`: a morphism of (odd) Lie algebras $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ such that

$$u_{\mathcal{M}} = \{J^\sharp_1(u), \cdot\} \text{ and } v_{\mathcal{M}} = \{J^\sharp_0(v), \cdot\},$$

  for $u \in \mathfrak{g}$ and $v \in \mathfrak{h}$.
- Dually, we can see a moment map $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ as a map of degree one manifolds $(J, J^\sharp) : \mathcal{M} \to (\overline{\mathfrak{g}}[-1])^*$.

- The constraint submanifold $\mathcal{C} := (J, J^\sharp)^{-1}(0)$ is the defined in terms of its sheaf of vanishing ideals
$$\mathcal{J} = \langle J_0^\sharp(v), J_1^\sharp(u)\rangle.$$

- The constraint submanifold $\mathcal{C} := (J, J^\sharp)^{-1}(0)$ is the defined in terms of its sheaf of vanishing ideals
$$\mathcal{J} = \langle J_0^\sharp(v), J_1^\sharp(u)\rangle.$$
- $J^\sharp : \bar{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ being a moment map is equivalent to $\mathcal{J}$ being a coisotropic ideal, that is, $\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}$.

- The constraint submanifold $\mathcal{C} := (J, J^\sharp)^{-1}(0)$ is the defined in terms of its sheaf of vanishing ideals
$$\mathcal{J} = \langle J_0^\sharp(v), J_1^\sharp(u)\rangle.$$

- $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ being a moment map is equivalent to $\mathcal{J}$ being a coisotropic ideal, that is, $\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}$.

graded $\overline{\mathfrak{g}}$-action $\Psi : \overline{\mathfrak{g}} \to \mathfrak{X}(\mathcal{M})$ with moment map $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$
$\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ with $J : M \to \mathfrak{h}^*$ equivariant, for $\mathfrak{h}$ a $\mathfrak{g}$-module

- Geometrically on M : we have the level set $C := J^{-1}(0) \subset M$ endowed with the involutive tangent distribution $\mathcal{D} := \langle J_1^\sharp(u)\rangle_{u\in\mathfrak{g}}$.

- For
$$N(\mathcal{J}) := \{f \in C^\infty(\mathcal{M}) \mid \{f, \mathcal{J}\} \subset \mathcal{J}\},$$
we have
$$\frac{N(\mathcal{J})}{\mathcal{J}} \cong C^\infty(\mathcal{C}_{red}),$$
when $\mathcal{C}_{red}$ exists.

- For
$$N(\mathcal{J}) := \{f \in C^\infty(\mathcal{M}) \mid \{f, \mathcal{J}\} \subset \mathcal{J}\},$$
we have
$$\frac{N(\mathcal{J})}{\mathcal{J}} \cong C^\infty(\mathcal{C}_{red}),$$
when $\mathcal{C}_{red}$ exists.

Theorem (Cattaneo-Zambon)

Let $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map for an infinitesimal action of $\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}$ on $\mathcal{M}$. Assume that $0 \in \mathfrak{h}^*$ is a regular value of $J : M \to \mathfrak{h}^*$ and that the action $\psi := J^\sharp|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{X}(M)$ on $C := J^{-1}(0)$ integrates to a free and proper action of a Lie group G, that is, assume that the pair (ψ, J) is regular. Then the corresponding degree one reduced space $\mathcal{C}_{red}$ exists and is naturally isomorphic to $T^*[1](C/G)$.

- A Poisson structure on M is a bivector $\pi \in \mathfrak{X}^2(M) = C_2^\infty(\mathcal{M})$ satisfying $\{\pi, \pi\} = 0$.

- A Poisson structure on M is a bivector $\pi \in \mathfrak{X}^2(M) = C^\infty_2(\mathcal{M})$ satisfying $\{\pi, \pi\} = 0$.
- Since
$$C^\infty(\mathcal{C}_{red}) \cong \frac{N(\mathcal{J})}{\mathcal{J}},$$
$\pi \in \mathfrak{X}^2(M)$ is said to be `reducible` iff $\pi \in N(\mathcal{J})$.

- A Poisson structure on M is a bivector $\pi \in \mathfrak{X}^2(M) = C_2^\infty(\mathcal{M})$ satisfying $\{\pi, \pi\} = 0$.
- Since
$$C^\infty(\mathcal{C}_{red}) \cong \frac{N(\mathcal{J})}{\mathcal{J}},$$
$\pi \in \mathfrak{X}^2(M)$ is said to be `reducible` iff $\pi \in N(\mathcal{J})$. In this case, we have $\pi_{red} \in \mathfrak{X}^2(C_{red})$.

- A Poisson structure on M is a bivector $\pi \in \mathfrak{X}^2(M) = C_2^\infty(\mathcal{M})$ satisfying $\{\pi, \pi\} = 0$.
- Since
$$C^\infty(\mathcal{C}_{red}) \cong \frac{N(\mathcal{J})}{\mathcal{J}},$$
$\pi \in \mathfrak{X}^2(M)$ is said to be `reducible` iff $\pi \in N(\mathcal{J})$. In this case, we have $\pi_{red} \in \mathfrak{X}^2(C_{red})$.
- For $\pi \in \mathfrak{X}^2(M) = C_2^\infty(\mathcal{M})$ to induce a Poisson structure on C_{red} it suffices that $\{\pi, \pi\} \in \mathcal{J}$ (*weak Poisson* – Lyakhovich-Sharapov, quasi-Poisson spaces).

- In non-graded terms, the condition $\pi \in N(\mathcal{J})$ gives:

- In non-graded terms, the condition $\pi \in N(\mathcal{J})$ gives:
  - $C^\infty(M)|_{\mathfrak{g}\text{-inv}} \subset C^\infty(M)$ is a Poisson subalgebra;
  - $\pi^\sharp(Ann(TC)) \subset \mathcal{D} := \langle \psi(u) \rangle_{u \in \mathfrak{g}}$.

- In non-graded terms, the condition $\pi \in N(\mathcal{J})$ gives:
  - $C^\infty(M)|_{\mathfrak{g}\text{-inv}} \subset C^\infty(M)$ is a Poisson subalgebra;
  - $\pi^\sharp(Ann(TC)) \subset \mathcal{D} := \langle \psi(u) \rangle_{u \in \mathfrak{g}}$.
- The Marsden-Weinstein theorem for the reduction of $\mathcal{M} := T^*[1]M$ implies the following `generalized version` of the classical Marsden-Weinstein theorem:

- In non-graded terms, the condition $\pi \in N(\mathcal{J})$ gives:
  - $C^\infty(M)|_{\mathfrak{g}\text{-inv}} \subset C^\infty(M)$ is a Poisson subalgebra;
  - $\pi^\sharp(Ann(TC)) \subset \mathcal{D} := \langle \psi(u) \rangle_{u \in \mathfrak{g}}$.
- The Marsden-Weinstein theorem for the reduction of $\mathcal{M} := T^*[1]M$ implies the following `generalized version` of the classical Marsden-Weinstein theorem:

Theorem (Cattaneo-Zambon)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the pair (ψ, J) is regular and compatible with the Poisson structure π. Then the quotient $C_{red} := C/G$ inherits a Poisson structure π_{red}.

- In non-graded terms, the condition $\pi \in N(\mathcal{J})$ gives:
  - $C^\infty(M)|_{\mathfrak{g}\text{-inv}} \subset C^\infty(M)$ is a Poisson subalgebra;
  - $\pi^\sharp(Ann(TC)) \subset \mathcal{D} := \langle \psi(u) \rangle_{u \in \mathfrak{g}}$.
- The Marsden-Weinstein theorem for the reduction of $\mathcal{M} := T^*[1]M$ implies the following `generalized version` of the classical Marsden-Weinstein theorem:

Theorem (Cattaneo-Zambon)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the pair (ψ, J) is regular and compatible with the Poisson structure π. Then the quotient $C_{red} := C/G$ inherits a Poisson structure π_{red}.

`The homological counterpart of this result will be derived from a homological model for the hamiltonian reduction of` $(\mathcal{M}, \{\cdot, \cdot\})$.

Let $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map describing the hamiltonian action of a graded Lie algebra $\overline{\mathfrak{g}} = \mathfrak{h}[1] \oplus \mathfrak{g}$.

Let $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map describing the hamiltonian action of a graded Lie algebra $\overline{\mathfrak{g}} = \mathfrak{h}[1] \oplus \mathfrak{g}$.

- Note $\mathcal{K}^\bullet := C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \bigwedge^\bullet \mathfrak{g}$ can be seen as the algebra of functions of $M \times T^*\mathfrak{g}^*[-1]$ and the BRST charge $Q_\mathfrak{g} \in \mathcal{K}^1$ as a function defining the homological vector field $\partial_{BRST} := \{Q_\mathfrak{g}, \cdot\}$.

Let $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map describing the hamiltonian action of a graded Lie algebra $\overline{\mathfrak{g}} = \mathfrak{h}[1] \oplus \mathfrak{g}$.

- Note $\mathcal{K}^\bullet := C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \bigwedge^\bullet \mathfrak{g}$ can be seen as the algebra of functions of $M \times T^*\mathfrak{g}^*[-1]$ and the BRST charge $Q_\mathfrak{g} \in \mathcal{K}^1$ as a function defining the homological vector field $\partial_{BRST} := \{Q_\mathfrak{g}, \cdot\}$.

1. Take

$$\mathcal{N} := \mathcal{M} \times T^*[1]\overline{\mathfrak{g}}^*[-1],$$

Let $J^\sharp : \bar{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map describing the hamiltonian action of a graded Lie algebra $\bar{\mathfrak{g}} = \mathfrak{h}[1] \oplus \mathfrak{g}$.

- Note $\mathcal{K}^\bullet := C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \bigwedge^\bullet \mathfrak{g}$ can be seen as the algebra of functions of $M \times T^*\mathfrak{g}^*[-1]$ and the BRST charge $Q_\mathfrak{g} \in \mathcal{K}^1$ as a function defining the homological vector field $\partial_{BRST} := \{Q_\mathfrak{g}, \cdot\}$.

1. Take
$$\mathcal{N} := \mathcal{M} \times T^*[1]\bar{\mathfrak{g}}^*[-1],$$
in such a way that
$$C^\infty(\mathcal{N}) = C^\infty(\mathcal{M}) \otimes C^\infty(T^*[1]\bar{\mathfrak{g}}^*[-1]) = C^\infty(\mathcal{M}) \otimes S^\bullet(\bar{\mathfrak{g}}^*[-1]) \otimes S^\bullet(\bar{\mathfrak{g}}).$$
Note $C^\infty(\mathcal{N})$ is endowed with a natural Poisson bracket $\{\cdot,\cdot\}$ of degree -1.

2. Let

$$A^{p,q} := C^{\infty}(\mathcal{M}) \otimes S^{p}(\bar{\mathfrak{g}}^{*}[-1]) \otimes S^{q}(\bar{\mathfrak{g}}) \subseteq C^{\infty}(\mathcal{N}).$$

2. Let

$$A^{p,q} := C^\infty(\mathcal{M}) \otimes S^p(\overline{\mathfrak{g}}^*[-1]) \otimes S^q(\overline{\mathfrak{g}}) \subseteq C^\infty(\mathcal{N}).$$

3. Define

$$\mathcal{A}^n := \bigoplus_{n=p-q} A^{p,q}.$$

2. Let

$$A^{p,q} := C^\infty(\mathcal{M}) \otimes S^p(\bar{\mathfrak{g}}^*[-1]) \otimes S^q(\bar{\mathfrak{g}}) \subseteq C^\infty(\mathcal{N}).$$

3. Define

$$\mathcal{A}^n := \bigoplus_{n=p-q} A^{p,q}.$$

In this way, $\mathcal{A} := C^\infty(\mathcal{N})$ becomes bigraded

2. Let
$$A^{p,q} := C^\infty(\mathcal{M}) \otimes S^p(\bar{\mathfrak{g}}^*[-1]) \otimes S^q(\bar{\mathfrak{g}}) \subseteq C^\infty(\mathcal{N}).$$

3. Define
$$\mathcal{A}^n := \bigoplus_{n=p-q} A^{p,q}.$$

In this way, $\mathcal{A} := C^\infty(\mathcal{N})$ becomes bigraded — $\mathcal{A}^{k,\ell}$ denote the subspace of $C^\infty(\mathcal{N})$ consisting of functions that have *total ghost number* k and *function degree* ℓ.

2. Let
$$A^{p,q} := C^\infty(\mathcal{M}) \otimes S^p(\bar{\mathfrak{g}}^*[-1]) \otimes S^q(\bar{\mathfrak{g}}) \subseteq C^\infty(\mathcal{N}).$$

3. Define
$$\mathcal{A}^n := \bigoplus_{n=p-q} A^{p,q}.$$

In this way, $\mathcal{A} := C^\infty(\mathcal{N})$ becomes bigraded — $\mathcal{A}^{k,\ell}$ denote the subspace of $C^\infty(\mathcal{N})$ consisting of functions that have *total ghost number* k and *function degree* ℓ.

$\mathcal{A} := C^\infty(\mathcal{N})$ with additional grading by *total ghost number* and the *BFV bracket* $\{\cdot,\cdot\}$.

2. Let
$$A^{p,q} := C^\infty(\mathcal{M}) \otimes S^p(\bar{\mathfrak{g}}^*[-1]) \otimes S^q(\bar{\mathfrak{g}}) \subseteq C^\infty(\mathcal{N}).$$

3. Define
$$\mathcal{A}^n := \bigoplus_{n=p-q} A^{p,q}.$$

   In this way, $\mathcal{A} := C^\infty(\mathcal{N})$ becomes bigraded — $\mathcal{A}^{k,\ell}$ denote the subspace of $C^\infty(\mathcal{N})$ consisting of functions that have *total ghost number* k and *function degree* ℓ.

$\mathcal{A} := C^\infty(\mathcal{N})$ with additional grading by *total ghost number* and the *BFV bracket* $\{\cdot,\cdot\}$.

4. The BRST charge: $Q_{\bar{\mathfrak{g}}} \in \mathcal{A}^{1,2}$ given by

2. Let

$$A^{p,q} := C^\infty(\mathcal{M}) \otimes S^p(\overline{\mathfrak{g}}^*[-1]) \otimes S^q(\overline{\mathfrak{g}}) \subseteq C^\infty(\mathcal{N}).$$

3. Define

$$\mathcal{A}^n := \bigoplus_{n=p-q} A^{p,q}.$$

In this way, $\mathcal{A} := C^\infty(\mathcal{N})$ becomes bigraded — $\mathcal{A}^{k,\ell}$ denote the subspace of $C^\infty(\mathcal{N})$ consisting of functions that have *total ghost number* k and *function degree* ℓ.

$\mathcal{A} := C^\infty(\mathcal{N})$ with additional grading by *total ghost number* and the *BFV bracket* $\{\cdot,\cdot\}$.

4. The BRST charge: $Q_{\overline{\mathfrak{g}}} \in \mathcal{A}^{1,2}$ given by

$$Q_{\overline{\mathfrak{g}}} = J_1^\sharp(u^i)u_i^* + J_0^\sharp(v^j)v_j^* - \frac{1}{2}c_k^{ij}\, u_i^*\, u_j^*\, u^k - d_p^{mn} u_m^*\, v_n^*\, v^p$$

2. Let
$$A^{p,q} := C^\infty(\mathcal{M}) \otimes S^p(\overline{\mathfrak{g}}^*[-1]) \otimes S^q(\overline{\mathfrak{g}}) \subseteq C^\infty(\mathcal{N}).$$

3. Define
$$\mathcal{A}^n := \bigoplus_{n=p-q} A^{p,q}.$$

In this way, $\mathcal{A} := C^\infty(\mathcal{N})$ becomes bigraded — $\mathcal{A}^{k,\ell}$ denote the subspace of $C^\infty(\mathcal{N})$ consisting of functions that have *total ghost number* k and *function degree* ℓ.

$\mathcal{A} := C^\infty(\mathcal{N})$ with additional grading by *total ghost number* and the *BFV bracket* $\{\cdot,\cdot\}$.

4. The BRST charge: $Q_{\overline{\mathfrak{g}}} \in \mathcal{A}^{1,2}$ given by
$$Q_{\overline{\mathfrak{g}}} = J_1^\sharp(u^i)u_i^* + J_0^\sharp(v^j)v_j^* - \frac{1}{2}c_k^{ij}\, u_i^*\, u_j^*\, u^k - d_p^{mn} u_m^*\, v_n^*\, v^p$$

(graded Jacobi identity for $\overline{\mathfrak{g}} \Leftrightarrow \{Q_{\overline{\mathfrak{g}}}, Q_{\overline{\mathfrak{g}}}\} = 0$).

Cohomology at $tgh = 0$

Cohomology at $tgh = 0$

Theorem (Bonechi-Cabrera-Zabzine)

Cohomology at $tgh = 0$

Theorem (Bonechi-Cabrera-Zabzine)

The cohomology of the complex $(\mathcal{A}, \{Q_{\overline{\mathfrak{g}}}, \cdot\})$ *at total ghost number zero is so that the natural map*

$$\Phi : H^{0,\bullet}_{\{Q_{\overline{\mathfrak{g}}}, \cdot\}}(\mathcal{A}) \longrightarrow \frac{N(\mathcal{J})}{\mathcal{J}}$$
$$[(x^{0,0} + x^{1,1} + \cdots)] \longmapsto \overline{x^{0,0}}$$

is an isomorphism of degree -1 *Poisson algebras.*

Cohomology at $tgh = 0$

Theorem (Bonechi-Cabrera-Zabzine)

The cohomology of the complex $(\mathcal{A}, \{Q_{\overline{\mathfrak{g}}}, \cdot\})$ at total ghost number zero is so that the natural map

$$\Phi : H^{0,\bullet}_{\{Q_{\overline{\mathfrak{g}}}, \cdot\}}(\mathcal{A}) \longrightarrow \frac{N(\mathcal{J})}{\mathcal{J}}$$
$$[(x^{0,0} + x^{1,1} + \cdots)] \longmapsto \overline{x^{0,0}}$$

is an isomorphism of degree -1 Poisson algebras.

- From the Marsden-Weinstein theorem for the reduction of $\mathcal{M} := T^*[1]M$, we know that $N(\mathcal{J})/\mathcal{J} \cong C^\infty(\mathcal{C}_{red})$, so the above result is a degree one version of the classical Kostant-Sternberg theorem.

Coupling a Poisson structure

A derived bracket construction

- If $\pi \in C_2^\infty(\mathcal{M})$ is reducible wrt to geometric data associated to $\mathcal{J}$, that is, $\pi \in N(\mathcal{J})$, then we can take a corresponding cohomology class $[\Pi] \in H^{0,2}_{\{Q_{\overline{\mathfrak{g}}},\cdot\}}$.

- If $\pi \in C_2^\infty(\mathcal{M})$ is reducible wrt to geometric data associated to $\mathcal{J}$, that is, $\pi \in N(\mathcal{J})$, then we can take a corresponding cohomology class $[\Pi] \in H^{0,2}_{\{Q_{\overline{\mathfrak{g}}},\cdot\}}$.
- For $[a], [b] \in H^{0,0}_{\{Q_{\overline{\mathfrak{g}}},\,\cdot\,\}}(\mathcal{A})$,

- If $\pi \in C_2^\infty(\mathcal{M})$ is reducible wrt to geometric data associated to $\mathcal{J}$, that is, $\pi \in N(\mathcal{J})$, then we can take a corresponding cohomology class $[\Pi] \in H^{0,2}_{\{Q_{\overline{\mathfrak{g}}},\cdot\}}$.
- For $[a],[b] \in H^{0,0}_{\{Q_{\overline{\mathfrak{g}}},\,\cdot\,\}}(\mathcal{A})$, the derived bracket

$$\{[a],[b]\}_\Pi := \{\{[\Pi],[a]\},[b]\} = [\{\{\Pi,a\},b\}]$$

- If $\pi \in C_2^\infty(\mathcal{M})$ is reducible wrt to geometric data associated to $\mathcal{J}$, that is, $\pi \in N(\mathcal{J})$, then we can take a corresponding cohomology class $[\Pi] \in H^{0,2}_{\{Q_{\overline{\mathfrak{g}}},\cdot\}}$.
- For $[a],[b] \in H^{0,0}_{\{Q_{\overline{\mathfrak{g}}},\,\cdot\,\}}(\mathcal{A})$, the derived bracket

$$\{[a],[b]\}_\Pi := \{\{[\Pi],[a]\},[b]\} = [\{\{\Pi,a\},b\}]$$

is a Poisson bracket.

- If $\pi \in C_2^\infty(\mathcal{M})$ is reducible wrt to geometric data associated to $\mathcal{J}$, that is, $\pi \in N(\mathcal{J})$, then we can take a corresponding cohomology class $[\Pi] \in H^{0,2}_{\{Q_{\overline{\mathfrak{g}}},\cdot\}}$.
- For $[a],[b] \in H^{0,0}_{\{Q_{\overline{\mathfrak{g}}},\,\cdot\,\}}(\mathcal{A})$, the derived bracket

$$\{[a],[b]\}_\Pi := \{\{[\Pi],[a]\},[b]\} = [\{\{\Pi,a\},b\}]$$

is a Poisson bracket.
- It turns out that

$$(H^{0,0}_{\{Q_{\overline{\mathfrak{g}}},\,\cdot\,\}}(\mathcal{A}),\{\cdot,\cdot\}_\Pi) \cong (C^\infty(C_{red}),\{\cdot,\cdot\}_{\pi_{red}}),$$

as Poisson algebras.

- Extended BRST charge:

- Extended BRST charge:

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \sum_{k\in\mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,\,2},$$

- `Extended BRST charge:`

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \sum_{k\in\mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Extended BRST charge:

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \sum_{k \in \mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,\,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Consider $\mathbb{Z}$-graded lagrangian submanifold $\mathcal{N}_{1,-1} \subset \mathcal{N}$ whose sheaf of functions is

$$C^{\infty}(\mathcal{N}_{1,-1}) := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h}.$$

Coupling a Poisson structure

Extended BRST charge

- Extended BRST charge:

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \sum_{k \in \mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Consider $\mathbb{Z}$-graded lagrangian submanifold $\mathcal{N}_{1,-1} \subset \mathcal{N}$ whose sheaf of functions is

$$C^{\infty}(\mathcal{N}_{1,-1}) := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h}.$$

- The extended charge $S^{(\infty)}$ induces a homotopy Poisson structure on $C^{\infty}(\mathcal{N}_{1,-1})$:

- `Extended BRST charge:`

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \sum_{k\in\mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Consider $\mathbb{Z}$-graded lagrangian submanifold $\mathcal{N}_{1,-1} \subset \mathcal{N}$ whose sheaf of functions is

$$C^\infty(\mathcal{N}_{1,-1}) := C^\infty(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h}.$$

- The extended charge $S^{(\infty)}$ induces a `homotopy Poisson structure` on $C^\infty(\mathcal{N}_{1,-1})$:

$$\ell_1(f_1) := \{Q_{\overline{\mathfrak{g}}}, f_1\}$$

- `Extended BRST charge:`

$$S^{(\infty)} = Q_{\mathfrak{g}} + \sum_{k\in\mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Consider $\mathbb{Z}$-graded lagrangian submanifold $\mathcal{N}_{1,-1} \subset \mathcal{N}$ whose sheaf of functions is

$$C^\infty(\mathcal{N}_{1,-1}) := C^\infty(M) \otimes \dot{\bigwedge} \mathfrak{g}^* \otimes \dot{\bigwedge} \mathfrak{h}.$$

- The extended charge $S^{(\infty)}$ induces a `homotopy Poisson structure` on $C^\infty(\mathcal{N}_{1,-1})$:

$$\ell_1(f_1) := \{Q_{\mathfrak{g}}, f_1\} \text{ and } \ell_2(f_1, f_2) := (-1)^{f_1}\{\{\Pi, f_1\}, f_2\}$$

- `Extended BRST charge:`

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \sum_{k \in \mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Consider $\mathbb{Z}$-graded lagrangian submanifold $\mathcal{N}_{1,-1} \subset \mathcal{N}$ whose sheaf of functions is

$$C^\infty(\mathcal{N}_{1,-1}) := C^\infty(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h}.$$

- The extended charge $S^{(\infty)}$ induces a `homotopy Poisson structure` on $C^\infty(\mathcal{N}_{1,-1})$:

$$\ell_1(f_1) := \{Q_{\overline{\mathfrak{g}}}, f_1\} \text{ and } \ell_2(f_1, f_2) := (-1)^{f_1}\{\{\Pi, f_1\}, f_2\}$$

and, in general, for $k \geq 3$,

- Extended BRST charge:

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \sum_{k\in\mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Consider $\mathbb{Z}$-graded lagrangian submanifold $\mathcal{N}_{1,-1} \subset \mathcal{N}$ whose sheaf of functions is

$$C^\infty(\mathcal{N}_{1,-1}) := C^\infty(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h}.$$

- The extended charge $S^{(\infty)}$ induces a homotopy Poisson structure on $C^\infty(\mathcal{N}_{1,-1})$:

$$\ell_1(f_1) := \{Q_{\overline{\mathfrak{g}}}, f_1\} \text{ and } \ell_2(f_1, f_2) := (-1)^{f_1}\{\{\Pi, f_1\}, f_2\}$$

and, in general, for $k \geq 3$, we let

$$\ell_k : C^\infty(\mathcal{N}_{1,-1})^{\otimes k} \to C^\infty(\mathcal{N}_{1,-1}) \quad \text{(of degree } 2-k)$$

- **Extended BRST charge:**

$$S^{(\infty)} = Q_{\mathfrak{g}} + \sum_{k \in \mathbb{N}} \Pi^{(-k)}, \quad \Pi^{(-k)} \in \mathcal{A}^{-k,2}, \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

- Consider $\mathbb{Z}$-graded lagrangian submanifold $\mathcal{N}_{1,-1} \subset \mathcal{N}$ whose sheaf of functions is

$$C^\infty(\mathcal{N}_{1,-1}) := C^\infty(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h}.$$

- The extended charge $S^{(\infty)}$ induces a homotopy Poisson structure on $C^\infty(\mathcal{N}_{1,-1})$:

$$\ell_1(f_1) := \{Q_{\mathfrak{g}}, f_1\} \text{ and } \ell_2(f_1, f_2) := (-1)^{f_1}\{\{\Pi, f_1\}, f_2\}$$

and, in general, for $k \geq 3$, we let

$$\ell_k : C^\infty(\mathcal{N}_{1,-1})^{\otimes k} \to C^\infty(\mathcal{N}_{1,-1}) \quad (\text{of degree } 2-k)$$

be defined by

$$\ell_k(f_1, \dots, f_k) := (-1)^{\epsilon}\{\dots\{\{\Pi^{(2-k)}, f_1\}, f_2\}, \dots, f_k\}.$$

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$,

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

  $$\ell_2(f, \ell_2(g,h)) - \left(\ell_2(\ell_2(f,g),h) + (-1)^{fg}\ell_2(g, \ell_2(f,h)) \right) = (\ell_3 \circ \ell_1^{\otimes 3} + \ell_1 \circ \ell_3)(f,g,h),$$

  where $\ell_1^{\otimes 3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes 3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ given by

  $$\ell_1^{\otimes 3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

$$\ell_2(f, \ell_2(g, h)) - \left(\ell_2(\ell_2(f, g), h) + (-1)^{fg} \ell_2(g, \ell_2(f, h)) \right) = (\ell_3 \circ \ell_1^{\otimes 3} + \ell_1 \circ \ell_3)(f, g, h),$$

where $\ell_1^{\otimes 3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes 3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ given by

$$\ell_1^{\otimes 3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

Remark

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

$$\ell_2(f, \ell_2(g, h)) - \left(\ell_2(\ell_2(f, g), h) + (-1)^{fg} \ell_2(g, \ell_2(f, h)) \right) = (\ell_3 \circ \ell_1^{\otimes_3} + \ell_1 \circ \ell_3)(f, g, h),$$

where $\ell_1^{\otimes_3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes_3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes_3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes_3}$ given by

$$\ell_1^{\otimes_3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

Remark

We have the identification $\mathcal{A}^{k,\ell} \cong \mathfrak{X}^{\ell-k,\,k}(\mathcal{N}_{1,-1})$,

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

$$\ell_2(f, \ell_2(g, h)) - \left(\ \ell_2(\ell_2(f, g), h) + (-1)^{fg} \ell_2(g, \ell_2(f, h)) \ \right) = (\ell_3 \circ \ell_1^{\otimes 3} + \ell_1 \circ \ell_3)(f, g, h),$$

where $\ell_1^{\otimes 3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes 3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ given by

$$\ell_1^{\otimes 3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

Remark

We have the identification $\mathcal{A}^{k,\ell} \cong \mathfrak{X}^{\ell-k,\, k}(\mathcal{N}_{1,-1})$, which shows $\mathcal{N} \cong T^*[1](\mathcal{N}_{1,-1})$

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

$$\ell_2(f, \ell_2(g, h)) - \left(\ell_2(\ell_2(f, g), h) + (-1)^{fg} \ell_2(g, \ell_2(f, h)) \right) = (\ell_3 \circ \ell_1^{\otimes 3} + \ell_1 \circ \ell_3)(f, g, h),$$

where $\ell_1^{\otimes 3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes 3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ given by

$$\ell_1^{\otimes 3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

Remark

We have the identification $\mathcal{A}^{k,\ell} \cong \mathfrak{X}^{\ell-k,\, k}(\mathcal{N}_{1,-1})$, which shows $\mathcal{N} \cong T^*[1](\mathcal{N}_{1,-1})$ – graded version of Weinstein lagrangian neighborhood theorem.

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

$$\ell_2(f, \ell_2(g, h)) - \left(\ell_2(\ell_2(f, g), h) + (-1)^{fg} \ell_2(g, \ell_2(f, h)) \right) = (\ell_3 \circ \ell_1^{\otimes 3} + \ell_1 \circ \ell_3)(f, g, h),$$

where $\ell_1^{\otimes 3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes 3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ given by

$$\ell_1^{\otimes 3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

Remark

We have the identification $\mathcal{A}^{k,\ell} \cong \mathfrak{X}^{\ell-k,\, k}(\mathcal{N}_{1,-1})$, which shows $\mathcal{N} \cong T^*[1](\mathcal{N}_{1,-1})$ – graded version of Weinstein lagrangian neighborhood theorem. Hence,

$$Q_{\overline{\mathfrak{g}}} \in \mathcal{A}^{1,2} \cong \mathfrak{X}^{1,\,1}(\mathcal{N}_{1,-1}), \;\; \Pi^{(-k)} \in \mathcal{A}^{-k,\,2} \cong \mathfrak{X}^{k+2,-k}(\mathcal{N}_{1,-1}),$$

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

$$\ell_2(f, \ell_2(g,h)) - \left(\ \ell_2(\ell_2(f,g),h) + (-1)^{fg}\ell_2(g, \ell_2(f,h)) \ \right) = (\ell_3 \circ \ell_1^{\otimes 3} + \ell_1 \circ \ell_3)(f,g,h),$$

where $\ell_1^{\otimes 3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes 3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ given by

$$\ell_1^{\otimes 3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

Remark

We have the identification $\mathcal{A}^{k,\ell} \cong \mathfrak{X}^{\ell-k,\,k}(\mathcal{N}_{1,-1})$, which shows $\mathcal{N} \cong T^*[1](\mathcal{N}_{1,-1})$ – graded version of Weinstein lagrangian neighborhood theorem. Hence,

$$Q_{\overline{\mathfrak{g}}} \in \mathcal{A}^{1,2} \cong \mathfrak{X}^{1,\,1}(\mathcal{N}_{1,-1}), \ \ \Pi^{(-k)} \in \mathcal{A}^{-k,\,2} \cong \mathfrak{X}^{k+2,-k}(\mathcal{N}_{1,-1}),$$

that is, $S^{(\infty)}$ gives a formal bivector on $\mathcal{N}_{1,-1}$

- From $\{Q, \Pi^{(-1)}\} + \frac{1}{2}\{\Pi, \Pi\} = 0$, we obtain

$$\ell_2(f, \ell_2(g,h)) - \big(\ell_2(\ell_2(f,g),h) + (-1)^{fg}\ell_2(g, \ell_2(f,h)) \big) = (\ell_3 \circ \ell_1^{\otimes 3} + \ell_1 \circ \ell_3)(f,g,h),$$

where $\ell_1^{\otimes 3} : C^\infty(\mathcal{N}_{1,-1})^{\otimes 3} \to C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ is the differential on $C^\infty(\mathcal{N}_{1,-1})^{\otimes 3}$ given by

$$\ell_1^{\otimes 3}(f \otimes g \otimes h) := \ell_1(f) \otimes g \otimes h + (-1)^f f \otimes \ell_1(g) \otimes h + (-1)^{f+g} f \otimes g \otimes \ell_1(h).$$

Remark

We have the identification $\mathcal{A}^{k,\ell} \cong \mathfrak{X}^{\ell-k,\,k}(\mathcal{N}_{1,-1})$, which shows $\mathcal{N} \cong T^*[1](\mathcal{N}_{1,-1})$ – graded version of Weinstein lagrangian neighborhood theorem. Hence,

$$Q_{\overline{\mathfrak{g}}} \in \mathcal{A}^{1,2} \cong \mathfrak{X}^{1,\,1}(\mathcal{N}_{1,-1}), \;\; \Pi^{(-k)} \in \mathcal{A}^{-k,\,2} \cong \mathfrak{X}^{k+2,-k}(\mathcal{N}_{1,-1}),$$

that is, $S^{(\infty)}$ gives a formal bivector on $\mathcal{N}_{1,-1}$ – P_∞-structure (Cattaneo-Felder); 0-shifted Poisson structure (Pridham).

Theorem (C.)

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module.

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ *be a Lie algebra action on a Poisson manifold* (M, π)*, and let* $J : M \to \mathfrak{h}^*$ *be a* $\mathfrak{g}$*-equivariant map, for* $\mathfrak{h}$ *a* $\mathfrak{g}$*-module. Assume that the reduction data* (ψ, J) *is regular and compatible with the Poisson structure* π*.*

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^\bullet_{\mathfrak{g},\mathfrak{h}} := C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \bigwedge^\bullet \mathfrak{h},$$

graded by total ghost number,

A homotopy Poisson structure from Poisson reduction

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h},$$

graded by total ghost number, admits a homotopy Poisson structure with differential $\partial : \mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} \to \mathcal{K}^{\bullet+1}_{\mathfrak{g},\mathfrak{h}}$ and a sequence of k-ary brackets $\{\cdot, \ldots, \cdot\}_k$, $k \geq 2$,

A homotopy Poisson structure from Poisson reduction

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h},$$

graded by total ghost number, admits a homotopy Poisson structure with differential $\partial : \mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} \to \mathcal{K}^{\bullet+1}_{\mathfrak{g},\mathfrak{h}}$ and a sequence of k-ary brackets $\{\cdot, \dots, \cdot\}_k$, $k \geq 2$, such that the Poisson algebra $(H^0_{\partial}(\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}}), \{\cdot,\cdot\}_2)$ is identified with the reduced Poisson algebra $(C^{\infty}(C_{red}), \{\cdot,\cdot\}_{\pi_{red}})$.

A homotopy Poisson structure from Poisson reduction

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h},$$

graded by total ghost number, admits a homotopy Poisson structure with differential $\partial : \mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} \to \mathcal{K}^{\bullet+1}_{\mathfrak{g},\mathfrak{h}}$ and a sequence of k-ary brackets $\{\cdot, \dots, \cdot\}_k$, $k \geq 2$, such that the Poisson algebra $(H^0_{\partial}(\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}}), \{\cdot,\cdot\}_2)$ is identified with the reduced Poisson algebra $(C^{\infty}(C_{red}), \{\cdot,\cdot\}_{\pi_{red}})$.

A homotopy Poisson structure from Poisson reduction

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^\bullet_{\mathfrak{g},\mathfrak{h}} := C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \bigwedge^\bullet \mathfrak{h},$$

graded by total ghost number, admits a homotopy Poisson structure with differential $\partial : \mathcal{K}^\bullet_{\mathfrak{g},\mathfrak{h}} \to \mathcal{K}^{\bullet+1}_{\mathfrak{g},\mathfrak{h}}$ and a sequence of k-ary brackets $\{\cdot, \dots, \cdot\}_k$, $k \geq 2$, such that the Poisson algebra $(H^0_\partial(\mathcal{K}^\bullet_{\mathfrak{g},\mathfrak{h}}), \{\cdot,\cdot\}_2)$ is identified with the reduced Poisson algebra $(C^\infty(C_{red}), \{\cdot,\cdot\}_{\pi_{red}})$.

Take $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ defined by the pair (ψ, J).

A homotopy Poisson structure from Poisson reduction

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h},$$

graded by total ghost number, admits a homotopy Poisson structure with differential $\partial : \mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} \to \mathcal{K}^{\bullet+1}_{\mathfrak{g},\mathfrak{h}}$ and a sequence of k-ary brackets $\{\cdot, \dots, \cdot\}_k$, $k \geq 2$, such that the Poisson algebra $(H^0_{\partial}(\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}}), \{\cdot,\cdot\}_2)$ is identified with the reduced Poisson algebra $(C^{\infty}(C_{red}), \{\cdot,\cdot\}_{\pi_{red}})$.

Take $J^{\sharp} : \overline{\mathfrak{g}}[-1] \to C^{\infty}(\mathcal{M})$ defined by the pair (ψ, J). Consider $(\mathcal{A}, \{Q_{\overline{\mathfrak{g}}}, \cdot\})$.

A homotopy Poisson structure from Poisson reduction

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^\bullet_{\mathfrak{g},\mathfrak{h}} := C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \bigwedge^\bullet \mathfrak{h},$$

graded by total ghost number, admits a homotopy Poisson structure with differential $\partial : \mathcal{K}^\bullet_{\mathfrak{g},\mathfrak{h}} \to \mathcal{K}^{\bullet+1}_{\mathfrak{g},\mathfrak{h}}$ and a sequence of k-ary brackets $\{\cdot, \dots, \cdot\}_k$, $k \geq 2$, such that the Poisson algebra $(H^0_\partial(\mathcal{K}^\bullet_{\mathfrak{g},\mathfrak{h}}), \{\cdot,\cdot\}_2)$ is identified with the reduced Poisson algebra $(C^\infty(C_{red}), \{\cdot,\cdot\}_{\pi_{red}})$.

Take $J^\sharp : \bar{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ defined by the pair (ψ, J). Consider $(\mathcal{A}, \{Q_{\bar{\mathfrak{g}}}, \cdot\})$. Regularity of (ψ, J) guarantees existence of $S^{(\infty)}$.

A homotopy Poisson structure from Poisson reduction

Theorem (C.)

Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action on a Poisson manifold (M, π), and let $J : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, for $\mathfrak{h}$ a $\mathfrak{g}$-module. Assume that the reduction data (ψ, J) is regular and compatible with the Poisson structure π. Then the $\mathbb{Z}$-graded algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h},$$

graded by total ghost number, admits a homotopy Poisson structure with differential $\partial : \mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}} \to \mathcal{K}^{\bullet+1}_{\mathfrak{g},\mathfrak{h}}$ and a sequence of k-ary brackets $\{\cdot, \dots, \cdot\}_k$, $k \geq 2$, such that the Poisson algebra $(H^0_{\partial}(\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{h}}), \{\cdot,\cdot\}_2)$ is identified with the reduced Poisson algebra $(C^{\infty}(C_{red}), \{\cdot,\cdot\}_{\pi_{red}})$.

Take $J^{\sharp} : \bar{\mathfrak{g}}[-1] \to C^{\infty}(\mathcal{M})$ defined by the pair (ψ, J). Consider $(\mathcal{A}, \{Q_{\bar{\mathfrak{g}}}, \cdot\})$. Regularity of (ψ, J) guarantees existence of $S^{(\infty)}$. Derived brackets on $\mathcal{K}_{\mathfrak{g},\mathfrak{h}} := C^{\infty}(\mathcal{N}_{1,-1})$.

Actions by dgla's

Actions by dgla's

Let $(\bar{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}, \delta)$ be a dgla concentrated in degrees -1 and 0.

Actions by dgla's

Let $(\bar{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}, \delta)$ be a dgla concentrated in degrees -1 and 0. For (M, π) a Poisson manifold, let $\Psi : (\bar{\mathfrak{g}}, \delta) \to (\mathfrak{X}(\mathcal{M}), [X_\pi, \cdot])$ be a morphism of dgla's, where $X_\pi := \{\pi, \cdot\}$, and let $J^\sharp : \bar{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map for this $\bar{\mathfrak{g}}$-action.

Actions by dgla's

Let $(\bar{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}, \delta)$ be a dgla concentrated in degrees -1 and 0. For (M, π) a Poisson manifold, let $\Psi : (\bar{\mathfrak{g}}, \delta) \to (\mathfrak{X}(\mathcal{M}), [X_\pi, \cdot\,])$ be a morphism of dgla's, where $X_\pi := \{\pi, \cdot\}$, and let $J^\sharp : \bar{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map for this $\bar{\mathfrak{g}}$-action.We have

$$\{\pi, J_1^\sharp(u)\} = 0, \quad u \in \mathfrak{g},$$

$$\{\pi, J_0^\sharp(v)\} = J_1^\sharp(\delta(v)), \quad v \in \mathfrak{h}.$$

Actions by dgla's

Let $(\bar{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}, \delta)$ be a dgla concentrated in degrees -1 and 0. For (M, π) a Poisson manifold, let $\Psi : (\bar{\mathfrak{g}}, \delta) \to (\mathfrak{X}(\mathcal{M}), [X_\pi, \cdot\,])$ be a morphism of dgla's, where $X_\pi := \{\pi, \cdot\}$, and let $J^\sharp : \bar{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map for this $\bar{\mathfrak{g}}$-action.We have

$$\{\pi, J_1^\sharp(u)\} = 0, \quad u \in \mathfrak{g},$$
$$\{\pi, J_0^\sharp(v)\} = J_1^\sharp(\delta(v)), \quad v \in \mathfrak{h}.$$

For

$$Q_{\bar{\mathfrak{g}}} = J_1^\sharp(u^i)u_i^* + J_0^\sharp(v^j)v_j^* - \frac{1}{2}c_k^{ij}\, u_i^*\, u_j^*\, u^k - d_p^{mn} u_m^*\, v_n^*\, v^p$$

Examples

Actions by dgla's

Let $(\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}, \delta)$ be a dgla concentrated in degrees -1 and 0. For (M, π) a Poisson manifold, let $\Psi : (\overline{\mathfrak{g}}, \delta) \to (\mathfrak{X}(\mathcal{M}), [X_\pi, \cdot\,])$ be a morphism of dgla's, where $X_\pi := \{\pi, \cdot\,\}$, and let $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map for this $\overline{\mathfrak{g}}$-action.We have

$$\{\pi, J^\sharp_1(u)\} = 0, \quad u \in \mathfrak{g},$$
$$\{\pi, J^\sharp_0(v)\} = J^\sharp_1(\delta(v)), \quad v \in \mathfrak{h}.$$

For

$$Q_{\overline{\mathfrak{g}}} = J^\sharp_1(u^i)u^*_i + J^\sharp_0(v^j)v^*_j - \frac{1}{2}c^{ij}_k\, u^*_i\, u^*_j\, u^k - d^{mn}_p u^*_m\, v^*_n\, v^p \;\text{ and }\; \Pi := \pi - a^i_j v^*_i u^j,$$

Actions by dgla's

Let $(\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}, \delta)$ be a dgla concentrated in degrees -1 and 0. For (M, π) a Poisson manifold, let $\Psi : (\overline{\mathfrak{g}}, \delta) \to (\mathfrak{X}(\mathcal{M}), [X_\pi, \cdot\,])$ be a morphism of dgla's, where $X_\pi := \{\pi, \cdot\,\}$, and let $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map for this $\overline{\mathfrak{g}}$-action.We have

$$\{\pi, J_1^\sharp(u)\} = 0, \quad u \in \mathfrak{g},$$
$$\{\pi, J_0^\sharp(v)\} = J_1^\sharp(\delta(v)), \quad v \in \mathfrak{h}.$$

For

$$Q_{\overline{\mathfrak{g}}} = J_1^\sharp(u^i)u_i^* + J_0^\sharp(v^j)v_j^* - \frac{1}{2}c_k^{ij}\, u_i^*\, u_j^*\, u^k - d_p^{mn} u_m^*\, v_n^*\, v^p \quad \text{and} \quad \Pi := \pi - a_j^i v_i^* u^j,$$

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \Pi \quad \text{satisfies} \quad \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

Actions by dgla's

Let $(\overline{\mathfrak{g}} := \mathfrak{h}[1] \oplus \mathfrak{g}, \delta)$ be a dgla concentrated in degrees -1 and 0. For (M, π) a Poisson manifold, let $\Psi : (\overline{\mathfrak{g}}, \delta) \to (\mathfrak{X}(\mathcal{M}), [X_\pi, \cdot\,])$ be a morphism of dgla's, where $X_\pi := \{\pi, \cdot\}$, and let $J^\sharp : \overline{\mathfrak{g}}[-1] \to C^\infty(\mathcal{M})$ be a moment map for this $\overline{\mathfrak{g}}$-action.We have

$$\{\pi, J_1^\sharp(u)\} = 0, \quad u \in \mathfrak{g},$$
$$\{\pi, J_0^\sharp(v)\} = J_1^\sharp(\delta(v)), \quad v \in \mathfrak{h}.$$

For

$$Q_{\overline{\mathfrak{g}}} = J_1^\sharp(u^i)u_i^* + J_0^\sharp(v^j)v_j^* - \frac{1}{2}c_k^{ij}\, u_i^*\, u_j^*\, u^k - d_p^{mn}u_m^*\, v_n^*\, v^p \;\text{ and }\; \Pi := \pi - a_j^i v_i^* u^j,$$

$$S^{(\infty)} = Q_{\overline{\mathfrak{g}}} + \Pi \;\text{ satisfies }\; \{S^{(\infty)}, S^{(\infty)}\} = 0.$$

It induces the structure of diff. graded Poisson algebra on

$$\mathcal{K}_{\mathfrak{g},\mathfrak{h}} = C^\infty(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{h}.$$

Examples

Lie bialgebra actions

Examples

Lie bialgebra actions

An action of the Lie bialgebra $(\mathfrak{g}, F)$ on the Poisson manifold (M, π) is a $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ for which

$$\{\pi, u^k_M\} = (F(u^k))_M = a^k_{ij} u^i_M u^j_M.$$

Examples

Lie bialgebra actions

An action of the Lie bialgebra $(\mathfrak{g}, F)$ on the Poisson manifold (M, π) is a $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ for which

$$\{\pi, u_M^k\} = (F(u^k))_M = a_{ij}^k u_M^i u_M^j.$$

In this case, the corresponding extended BRST charge induces a homotopy Poisson structuce on $\mathcal{K}_{\mathfrak{g}} = C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^*$

Examples

Lie bialgebra actions

An action of the Lie bialgebra $(\mathfrak{g}, F)$ on the Poisson manifold (M, π) is a $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ for which

$$\{\pi, u^k_M\} = (F(u^k))_M = a^k_{ij} u^i_M u^j_M.$$

In this case, the corresponding extended BRST charge induces a homotopy Poisson structuce on $\mathcal{K}_{\mathfrak{g}} = C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^*$ and its first two terms are

$$Q_{\mathfrak{g}} = u^i_M u^*_i - \frac{1}{2} c^{ij}_k u^*_i u^*_j u^k$$

Lie bialgebra actions

An action of the Lie bialgebra $(\mathfrak{g}, F)$ on the Poisson manifold (M, π) is a $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ for which

$$\{\pi, u^k_M\} = (F(u^k))_M = a^k_{ij} u^i_M u^j_M.$$

In this case, the corresponding extended BRST charge induces a homotopy Poisson structuce on $\mathcal{K}_\mathfrak{g} = C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^*$ and its first two terms are

$$Q_\mathfrak{g} = u^i_M u^*_i - \frac{1}{2} c^{ij}_k u^*_i u^*_j u^k \quad \text{and} \quad \Pi = \pi + a^j_{ik} u^i_M u^*_j u^k.$$

Lie bialgebra actions

An action of the Lie bialgebra $(\mathfrak{g}, F)$ on the Poisson manifold (M, π) is a $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ for which

$$\{\pi, u_M^k\} = (F(u^k))_M = a_{ij}^k u_M^i u_M^j.$$

In this case, the corresponding extended BRST charge induces a homotopy Poisson structuce on $\mathcal{K}_\mathfrak{g} = C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^*$ and its first two terms are

$$Q_\mathfrak{g} = u_M^i u_i^* - \frac{1}{2} c_k^{ij} u_i^* u_j^* u^k \quad \text{and} \quad \Pi = \pi + a_{ik}^j u_M^i u_j^* u^k.$$

(Here, we consider $\mathfrak{h} = \{0\}$, so $J \equiv 0$.)

Proposition (C.)

Let (M, π) be a Poisson manifold, and let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be an action of a Lie bialgebra $(\mathfrak{g}, F)$. Then the graded algebra

$$\mathcal{K}_{\mathfrak{g}} := C^{\infty}(M) \otimes \bigwedge^{\bullet} \mathfrak{g}^*$$

admits a homotopy Poisson structure with the 2-ary bracket given by

$$\ell_2(f, g) = \{\{\pi, f\}, g\}, \quad f, g \in C^{\infty}(M),$$

$$\ell_2(f, u^*) = \{u^i_M, f\}[u^*_i, u^*]^*, \quad f \in C^{\infty}(M), \; u^* \in \mathfrak{g}^*,$$

$$\ell_2(u^*_1, u^*_2) = 0, \quad u^*_1, u^*_2 \in \mathfrak{g}^*.$$

Examples

Quasi-Poisson spaces

Quasi-Poisson spaces

Given a quasi-Lie bialgebra $(\mathfrak{g}, F, \chi)$, a quasi-Poisson $\mathfrak{g}$-space is a manifold M endowed an $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and a bivector field $\pi \in \mathfrak{X}^2(M)$

Quasi-Poisson spaces

Given a quasi-Lie bialgebra $(\mathfrak{g}, F, \chi)$, a quasi-Poisson $\mathfrak{g}$-space is a manifold M endowed an $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and a bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying

$$\{\pi, u_M^k\} = (F(u^k))_M = a_{ij}^k u_M^i u_M^j \quad \text{and} \quad \frac{1}{2}\{\pi, \pi\} = \chi_M.$$

Examples

Quasi-Poisson spaces

Given a quasi-Lie bialgebra $(\mathfrak{g}, F, \chi)$, a quasi-Poisson $\mathfrak{g}$-space is a manifold M endowed an $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and a bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying

$$\{\pi, u_M^k\} = (F(u^k))_M = a_{ij}^k u_M^i u_M^j \quad \text{and} \quad \frac{1}{2}\{\pi, \pi\} = \chi_M.$$

In this case, the corresponding extended BRST charge induces a homotopy Poisson structure on $\mathcal{K}_{\mathfrak{g}} = C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^*$

Examples

Quasi-Poisson spaces

Given a quasi-Lie bialgebra $(\mathfrak{g}, F, \chi)$, a quasi-Poisson $\mathfrak{g}$-space is a manifold M endowed an $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and a bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying

$$\{\pi, u_M^k\} = (F(u^k))_M = a_{ij}^k u_M^i u_M^j \quad \text{and} \quad \frac{1}{2}\{\pi, \pi\} = \chi_M.$$

In this case, the corresponding extended BRST charge induces a homotopy Poisson structure on $\mathcal{K}_\mathfrak{g} = C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^*$ and its first three terms are

$$Q_\mathfrak{g} = u_M^i u_i^* - \frac{1}{2} c_k^{ij} u_i^* u_j^* u^k, \quad \Pi = \pi + a_{ik}^j u_M^i u_j^* u^k$$

Quasi-Poisson spaces

Given a quasi-Lie bialgebra $(\mathfrak{g}, F, \chi)$, a quasi-Poisson $\mathfrak{g}$-space is a manifold M endowed an $\mathfrak{g}$-action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and a bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying

$$\{\pi, u_M^k\} = (F(u^k))_M = a_{ij}^k u_M^i u_M^j \quad \text{and} \quad \frac{1}{2}\{\pi, \pi\} = \chi_M.$$

In this case, the corresponding extended BRST charge induces a homotopy Poisson structure on $\mathcal{K}_\mathfrak{g} = C^\infty(M) \otimes \bigwedge^\bullet \mathfrak{g}^*$ and its first three terms are

$$Q_\mathfrak{g} = u_M^i u_i^* - \frac{1}{2} c_k^{ij} u_i^* u_j^* u^k, \quad \Pi = \pi + a_{ik}^j u_M^i u_j^* u^k$$

$$\text{and} \quad \Pi^{-1} = \frac{1}{3}\psi(\iota_{u_i^*}\chi)u^i + \cdots .$$

Hamiltonian quasi-Poisson spaces

Recall that a G-manifold M endowed with a *invariant bivector* $\pi \in \mathfrak{X}^2(M)$ is said to be a hamiltonian quasi-Poisson space if

$$\{\pi, \pi\} = \phi_M := \frac{1}{12} \sum_{i,j,k} \langle u^i, [u^j, u^k] \rangle (u^i \wedge u^j \wedge u^j)_M,$$

and if there exists an equivariant map $\Phi : M \to G$ for which we have the moment map condition

$$\pi^\sharp(d(\Phi^* f)) = \frac{1}{2} \sum_k \Phi^*\big((u_L^k + u_R^k) f\big) u_M^k, \quad f \in C^\infty(G). \tag{3}$$

In this context, if the identity $e \in G$ is a regular value of $\Phi : M \to G$ and the action of G along $\Phi^{-1}(e)$ is free and proper, then the quotient $\Phi^{-1}(e)/G$ inherits a Poisson structure (Alekseev–Kosmann-Schwarzbach–Meinrenken).

Proposition (C.)

Proposition (C.)

Let (M, π) be a hamiltonian quasi-Poisson space with moment map $\Phi : M \to G$.

Proposition (C.)

Let (M, π) *be a hamiltonian quasi-Poisson space with moment map* $\Phi : M \to G$*. Assume that the identity* $e \in G$ *is a regular value of* $\Phi : M \to G$ *and that the* G*-action on* $\Phi^{-1}(e)$ *is free and proper.*

Proposition (C.)

Let (M, π) *be a hamiltonian quasi-Poisson space with moment map* $\Phi : M \to G$*. Assume that the identity* $e \in G$ *is a regular value of* $\Phi : M \to G$ *and that the* G*-action on* $\Phi^{-1}(e)$ *is free and proper. Let* $\mathcal{U} \subset G$ *be neighborhood of* $e \in G$ *where* $\exp : \mathfrak{g} \to G$ *is diffeomorphism and set* $M_{\mathcal{U}} := \Phi^{-1}(\mathcal{U})$*.*

Proposition (C.)

Let (M, π) be a hamiltonian quasi-Poisson space with moment map $\Phi : M \to G$. Assume that the identity $e \in G$ is a regular value of $\Phi : M \to G$ and that the G-action on $\Phi^{-1}(e)$ is free and proper. Let $\mathcal{U} \subset G$ be neighborhood of $e \in G$ where $\exp : \mathfrak{g} \to G$ is diffeomorphism and set $M_{\mathcal{U}} := \Phi^{-1}(\mathcal{U})$. Then the algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{g}} := C^{\infty}(M_{\mathcal{U}}) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{g}$$

admits a homotopy Poisson structure

Proposition (C.)

Let (M,π) be a hamiltonian quasi-Poisson space with moment map $\Phi : M \to G$. Assume that the identity $e \in G$ is a regular value of $\Phi : M \to G$ and that the G-action on $\Phi^{-1}(e)$ is free and proper. Let $\mathcal{U} \subset G$ be neighborhood of $e \in G$ where $\exp : \mathfrak{g} \to G$ is diffeomorphism and set $M_{\mathcal{U}} := \Phi^{-1}(\mathcal{U})$. Then the algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{g}} := C^{\infty}(M_{\mathcal{U}}) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{g}$$

admits a homotopy Poisson structure for which we have

$$(H^0_{\partial}(\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{g}}), \{\cdot,\cdot\}_2) \cong \left(C^{\infty}\left(\frac{\Phi^{-1}(e)}{G}\right), \pi_{red}\right).$$

Proposition (C.)

Let (M, π) be a hamiltonian quasi-Poisson space with moment map $\Phi : M \to G$. Assume that the identity $e \in G$ is a regular value of $\Phi : M \to G$ and that the G-action on $\Phi^{-1}(e)$ is free and proper. Let $\mathcal{U} \subset G$ be neighborhood of $e \in G$ where $\exp : \mathfrak{g} \to G$ is diffeomorphism and set $M_{\mathcal{U}} := \Phi^{-1}(\mathcal{U})$. Then the algebra

$$\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{g}} := C^{\infty}(M_{\mathcal{U}}) \otimes \bigwedge^{\bullet} \mathfrak{g}^* \otimes \bigwedge^{\bullet} \mathfrak{g}$$

admits a homotopy Poisson structure for which we have

$$(H^0_{\partial}(\mathcal{K}^{\bullet}_{\mathfrak{g},\mathfrak{g}}), \{\cdot,\cdot\}_2) \cong \left(C^{\infty}\left(\frac{\Phi^{-1}(e)}{G}\right), \pi_{red}\right).$$

This result provides a BFV model for hamiltonian quasi-Poisson reduction.

Thank you!