Odd and generalised Wilson surfaces

Olga Chekeres

University of l'Aquila

Based on joint work with Vladimir Salnikov

"Cohomology in algebra, geometry, physics and statistics",
Prague,
Institute of Mathematics of ASCR,
19 March 2025

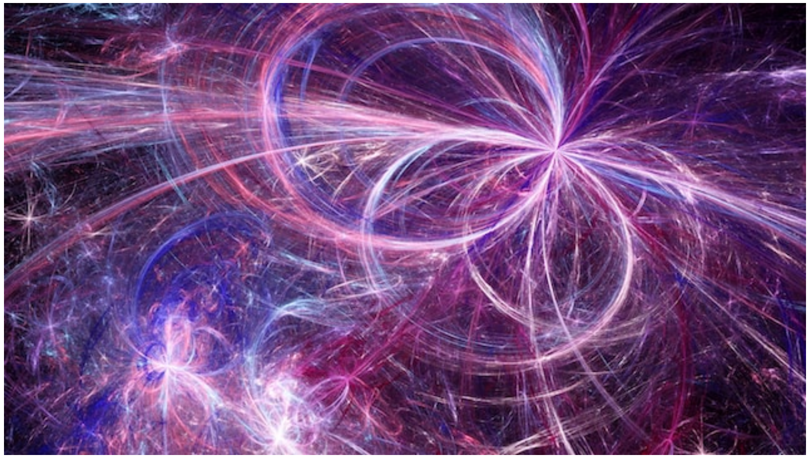
Thank you

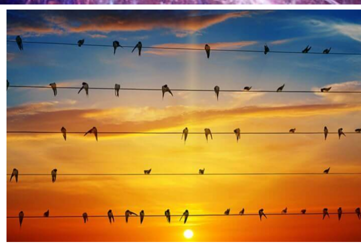
for your attention!

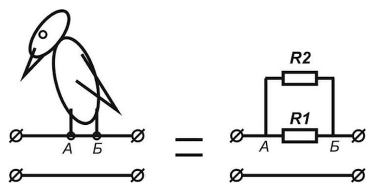
Děkuji za pozornost!

Gauge theories

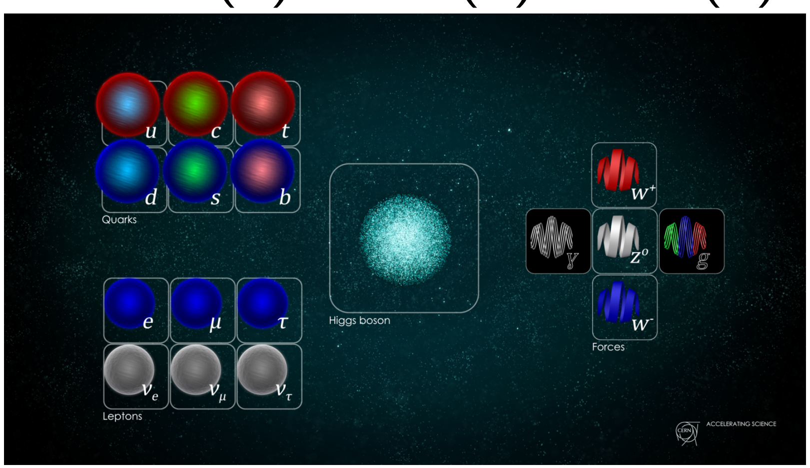
Électromagnetism: $G = U(1)$



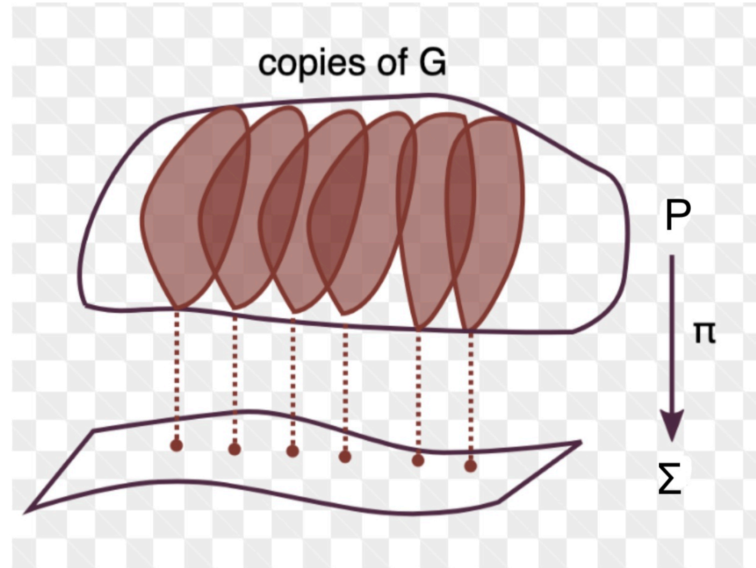




Standard model:
$G = U(1) \times SU(2) \times SU(3)$



Geometric picture – principal G-bundle



G — compact connected Lie group

$\mathfrak{g} = Lie(G)$ its Lie algebra

Tr – invariant product on $\mathfrak{g}$

gauge connection $A \in \Omega^1(P, \mathfrak{g})$;

curvature:

$F = dA + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g})$.

Nonlocal observables

Wilson loop observable

A Wilson line is defined by

- the holonomy of the gauge field A along a closed curve Γ, embedded in a manifold N,
- finite dimensional representation R of G.

Wilson line formula:

$$W_\Gamma^R(A) = \mathrm{Tr}_R\, P \exp\left(\int_\Gamma A^R\right),$$

The gauge field takes values in the Lie algebra $\mathfrak{g}$ of G:

$$A^R = \sum_{a,i} A_i^a\, t_a^R\, dx^i.$$

The gauge invariance is guaranteed by taking the trace in the representation R.

Alekseev-Faddeev-Shatashvili presentation of Wilson Line

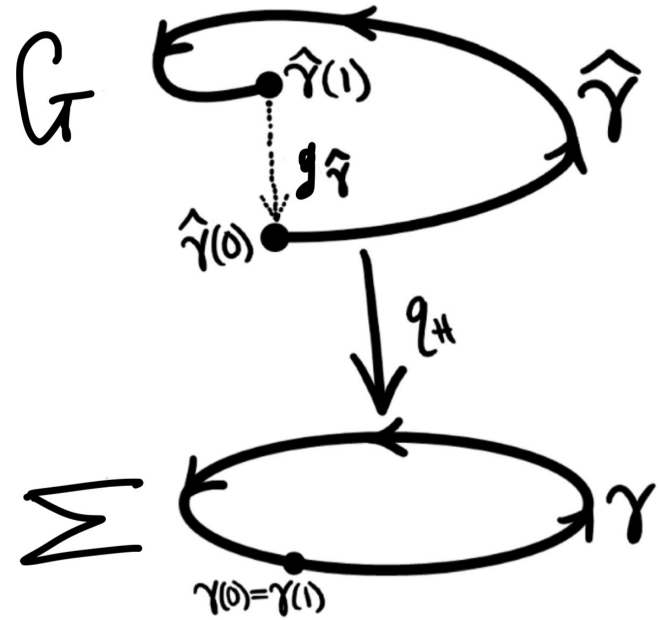
Involves a path integral quantization of coadgoint orbits of G.

- An irreducible finite dimensional representation is uniquely determined by its highest weight $\lambda \in \mathfrak{h}^*$, $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.
- Associate to λ the orbit of the coadjoint action in the space of $\mathfrak{g}^*$.
- Denote the coadjoint action by $Ad_g^*(\lambda) = g\lambda g^{-1}$.

The coadjoint orbit:

$$\mathcal{O}_\lambda = \{g\lambda g^{-1} | \lambda \in \mathfrak{g}^*, g \in G\}.$$

Nonlocal observables. Wilson lines and surfaces



$g : \gamma \to G, \quad A \in \Omega^1(P, \mathfrak{g})$
$\lambda \in \mathfrak{h}^*$ – the highest weight of the representation R
$\mathcal{O}_\lambda = \{g\lambda g^{-1} \in \mathfrak{g}^* | g \in G, \lambda \in \mathfrak{g}^*\}$.
$b : \gamma \to \mathfrak{g}^*$ is a field s.t.
$b(t) = g(t)\lambda g(t)^{-1}$.

$$W_\gamma^R = Tr_R Pexp\Big(\int_\gamma A\Big) = \int \mathcal{D}g e^{iS_\lambda(g,A)},$$

where $S_{WL}(A, g, b) = \int_\gamma Trb\Big(dgg^{-1} + A\Big)$ is WL action functional.

$S_{WS}(A, g) = \int_\Sigma Trb\Big(F_A + (d_A gg^{-1})^2\Big)$ is the bulk / WS functional

Important result: S_{WS} is defined by an equivariant extension of Kirillov–Kostant–Souriau symplectic form on $\mathcal{O}_\lambda$

Interpretation:

$$\Omega^*_G(\mathcal{O}_\lambda) = \{\alpha \in \Omega^*(\mathcal{O}_\lambda) \otimes W_G \mid \mathcal{L}_\xi \alpha = 0,\ \iota_\xi \alpha = 0\}$$

Space of equivariant forms on $\mathcal{O}_\lambda$

- $H^*_G(\mathcal{O}_\lambda) = H^*(\Omega^*_G(\mathcal{O}_\lambda),\ d\otimes 1 + 1\otimes d)$

The Weil model of equivariant cohomology on $\mathcal{O}_\lambda$.

$$\boxed{\mathrm{Tr}\, b\,(F_A - (d_A g g^{-1})^2)}$$ – is an equivariant cocycle

$$H^*_G(\mathcal{O}_\lambda) \simeq \frac{\mathcal{O}_\lambda \times EG}{G}$$

Recall: Poisson σ-model

- Consider a Poisson manifold (M, π)
- with Poisson structure $\pi = \frac{1}{2}\,\pi^{ij}\,\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$.

Poisson σ-model:

$$S^\pi(X, \alpha) = \int_\Sigma \left(\alpha_i dX^i + \frac{1}{2}\pi^{ij}(X)\alpha_i \wedge \alpha_j \right),$$

where $X^i = x^i \circ X$ are components of the map $X : \Sigma \to M$,
α_i are 1-forms on Σ representing gauge fields of the Poisson σ-model.

- In case π^{ij} is invertible, (M, π) is also symplectic with $\omega_{ij} = (\pi^{-1})^{ji}$.

Recall: Poisson σ-model on a coadjoint orbit

- Let the target space be $\mathcal{O}_\lambda$,
- then the action

$$S_{\varpi_O} = -\int_\Sigma \mathrm{Tr}\,\lambda(g^{-1}dg)^2 = -\int_\Sigma \mathrm{Tr}\,b(dgg^{-1})^2.$$

not equal, but equivalent actions

Poisson σ-model on $\mathcal{O}_\lambda$:

$$S^\pi(b,\alpha) = \int_\Sigma \mathrm{Tr}\,b(d\alpha + \alpha^2),$$

where $\alpha \in \Omega^1(\Sigma, \mathfrak{g})$ is the auxiliary gauge field.

$\int \mathcal{D}b\,\mathcal{D}\alpha\,\mathcal{D}g\, e^{iS}$

- Remark:

$$\int_\Sigma \mathrm{Tr}\,b(d\alpha + \alpha^2) = \int_\Sigma \mathrm{Tr}\,b\left((dgg^{-1} + \alpha)^2 - (dgg^{-1})^2\right).$$

Poisson σ-model for a Wilson surface

- View Wilson surface action as a version of the action S_{ϖ_O} interacting with the external gauge field A.

Poisson σ-model version of the action

$$S_\sigma(b, A, \alpha) = \int_\Sigma \mathrm{Tr}\, b \left(F_A + (d_A g g^{-1} + \alpha)^2 - (d_A g g^{-1})^2\right).$$

Poisson σ-model as a BF theory

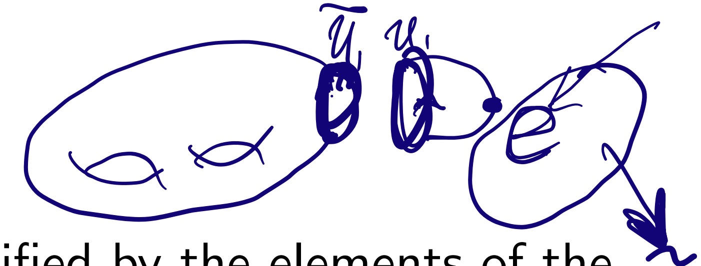
$$S_\sigma(b, A, \alpha) = \int_\Sigma \mathrm{Tr}\, b\, F_{A+\alpha},$$

where the field b is constrained and $A + \alpha$ is a new connection on P.

constrained to $\mathcal{O}_\lambda$

$b : \Sigma \to \mathcal{O}_\lambda \subset \mathfrak{g}^*$

Recall: Canonical quantization of 2-dimensional BF theory



- Principal G-bundles over Σ are classified by the elements of the fundamental group of G: $\pi_1(G) \cong \Gamma \subset Z(\tilde{G})$.

$G \simeq \tilde{G}/\Gamma$

- Consider a surface with one boundary component. Gluing this puncture to an infinitesimal disc is described by identifying $U = C_i$, where $C_i \in \Gamma$ is a central element of $\tilde{G}$.

The formula for a closed surface:

- The contribution to the partition function of each class $[P]$ of a principal G-bundle over the surface:

$$Z_{BF}^{\Sigma}(C_i) = \sum_R d_R^{1-2g} \chi_R(C_i).$$

Migdal, Witten

Wilson surface theory

Theory on a closed surface

The formula of the Wilson surface theory for a closed surface for a class $[P]$ of principal bundles $P \to \Sigma$:

$$Z_{WS}^{\Sigma}(C_i, \lambda) = \frac{\chi_\lambda(C_i)}{d_\lambda}. \quad (10)$$

holonomy of A+d

$G = \tilde{G}/\Gamma$

$\tilde{G}$ = covering group (simply connected)

$\Gamma \subset Z(\tilde{G})$ subgroup of the center of $\tilde{G}$

$\pi_1(G) \simeq \Gamma$

$C_i \in \Gamma$ central elements in $\tilde{G}$

Result 3: Topological interactions

The presence of a Wilson surface modifies the partition function of the background theory multiplying by a phase $e^{i\varphi_\gamma} = \frac{\chi_\lambda(C_\gamma)}{d_\lambda}$ the individual contributions for each class of principal bundles:

$$Z^{interact} = \sum_{\gamma \in \pi_1(G)} Z^{backgr}(C_\gamma) \cdot e^{i\varphi_\gamma}.$$

Because the world is an odd place..

- We add odd degrees of freedom to the structure group
- We add odd degrees of freedom to the target
- We study linear super Poisson sigma model with super coadjoint orbit as a target

Super Wilson surface action functional

$$S = \int_{\Sigma} Tr\Big(\mathbf{b}\mathbf{F}_{\mathbf{A}+\mathbf{a}}\Big).$$

$G = G_0 \ltimes \Pi\mathfrak{g}_0$ is a matrix Lie supergroup , $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.
The fields are now superfields.
The gauge potential is $\mathbf{A} = A_i^{a\xi} dx^i \otimes T_a \otimes e_\xi \in \Omega^1(P, \mathfrak{g}_0 \oplus \mathfrak{g}_1)$.
The field $\mathbf{b} : \Sigma \to \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*$
$\mathbf{b} = (g, \alpha)(X, \beta)(g, \alpha)^{-1}$,
where $(g, \alpha) \in G$,
$(X, \beta) = \lambda \in \mathfrak{h}_0^* \oplus \mathfrak{h}_1^*$ is the orbit representative with
$\mathfrak{h}_0^* \subset \mathfrak{g}_0^*$ being the dual to the Cartan subalgebra of $\mathfrak{g}_0$,
$\mathfrak{h}_1^* = \Pi\mathfrak{h}_0^*$ being the odd extension of $\mathfrak{h}_0^*$;
$\mathbf{a} \in \Omega^1_{hor}(P, \mathfrak{g})^G$, as before, is an auxiliary field,
Tr is invariant bilinear product on the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Example $G = UOSp(1|2)$

The algebra $\mathfrak{uosp}(1|2)$: the even (bosonic) generators h, b_+, b_- are those of $\mathfrak{su}(2)$,
the odd (fermionic) generators f_+, f_- are the basis of the odd subspace $\Pi\mathfrak{p}_\circ$,
where $\mathfrak{p}_\circ$ is the invariant compliment of the Cartan subalgebra $\mathfrak{h}_\circ \subset \mathfrak{g}_0$ in the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{p}_0$,
h is the generator of $\mathfrak{h}_0$.
Then $\mathfrak{uosp}(1|2) = \mathfrak{su}(2) \oplus \Pi\mathfrak{p}_\circ$. $su(2) = u(1) \oplus p_0$
The commutation relations read:

$$[h, b_+] = b_+, \quad [h, b_-] = -b_-, \quad [b_+, b_-] = 2h, \quad [h, f_+] = \tfrac{1}{2} f_+,$$
$$[h, f_-] = -\tfrac{1}{2} f_-, \quad [b_-, f_+] = f_-, \quad [b_+, f_-] = f_+, [b_+, f_+] = 0,$$
$$[b_-, f_-] = 0, \; [f_+, f_+] = \tfrac{1}{2} b_+, \quad [f_-, f_-] = -\tfrac{1}{2} b_-, \quad [f_+, f_-] = -\tfrac{1}{2} h.$$

Example $G = UOSp(1|2)$

We use the fact that for $UOSp(1|2)$, $\mathfrak{g} \cong \mathfrak{g}^*$ and identify adjoint and coadjoint orbits.

It is enough to consider the orbit representative $(X, 0) \in \mathfrak{h} = \mathfrak{h}_0 \oplus 0$ which is purely even, since Cartan subalgebra of $\mathfrak{g}_0$ does not have an odd counterpart.

The stabilizer of $(X, 0)$ is purely even: $\mathcal{H} = U(1)$.
And the orbit is

$$\mathcal{O}_{(X,0)} = SU(2) \ltimes \Pi\mathfrak{p}_{\mathfrak{o}}/U(1) \cong S^2 \times \Pi\mathbb{R}^2.$$

Example $G = UOSp(1|2)$

Then the partition function of the Wilson surface labeled by the element (X, β) for a particular equivalence class of principal bundles $P \to \Sigma$, defined by $\gamma \in \pi_1(G)$ is given by:

$$Z(C_\gamma, (X, \beta)) = \frac{\chi_{(X,\beta)}(C_\gamma)}{sD_{(X,\beta)}},$$

where $\chi_{(X,\beta)}(C_\gamma) = sTr(C_\gamma)$ is a value of the character $\chi_{(X,\beta)}$ on the element C_γ in the representation $R_{(X,\beta)}$, corresponding to the orbit element (X, β), and $sD_{(X,\beta)}$ is the "super dimension" of the matrix $R_{(X,\beta)}(C_\gamma)$ (i.e. number of nontrivial diagonal elements of the even block minus the number of diagonal elements in the odd block in its normal form).

For $G = UOSp(1|2)$ the highest weight $(X, 0) \in \mathfrak{h}^*$ defines an irreducible finite-dimensional representation. In this case the target space of the theory is given by the coadjoint orbit $\mathcal{O}_{(X,0)}$ passing through the point $(X, 0) \in \mathfrak{g}^*$.

For $G = UOSp(1|2)$, since there exists only the trivial class of principal G-bundles over a closed surface Σ, the partition function of the Wilson surface is

$$Z(e, (X, 0)) = \frac{\chi_{(X,0)}(e)}{sD_{(X,0)}} = \frac{sTr_{(X,0)}(e)}{sD_{(X,0)}} = \frac{sD_{(X,0)}}{sD_{(X,0)}} = 1,$$

just like in non-supersymmetric case.

Bosonization conjecture

For $G = UOSp(1|2)/\mathbb{Z}_2 = SO(3) \ltimes \Pi\mathfrak{p}_0$ there are two classes of principal G-bundles over a closed surface Σ. The partition function of the Wilson surface for the trivial class is

$$Z_{triv} = Z(e,(X,0)) = \frac{\chi_{(X,0)}(e)}{sD_{(X,0)}} = \frac{sTr_{(X,0)}(e)}{sD_{(X,0)}} = \frac{sD_{(X,0)}}{sD_{(X,0)}} = 1.$$

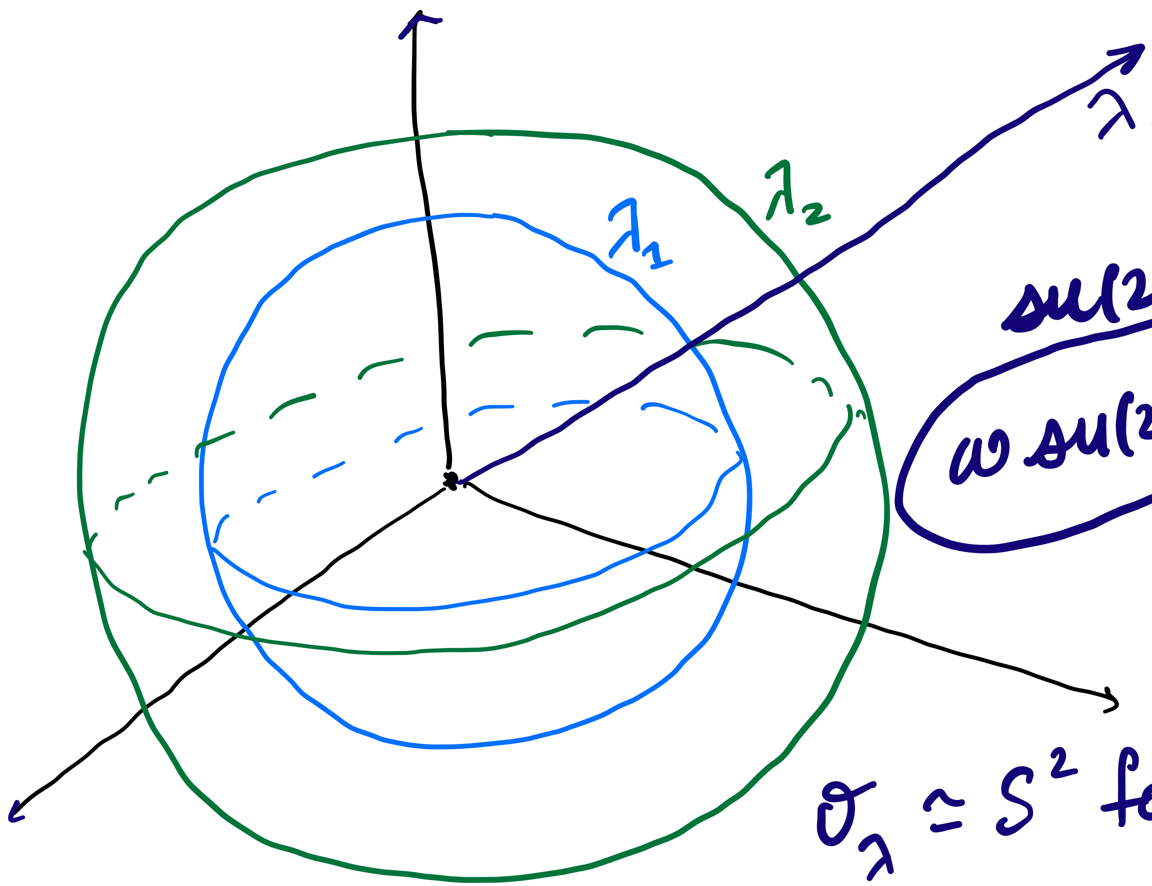
The partition function of the Wilson surface for the nontrivial class:

$$Z_{nontriv} = Z(-e,(X,0)) = \frac{\chi_{(X,0)}(-e)}{sD_{(X,0)}} = \frac{sTr_{(X,0)}(-e)}{sD_{(X,0)}} = \frac{-sD_{(X,0)}}{sD_{(X,0)}} = -1.$$

This result agrees with the (non supersymmetric) case of $G = SU(2)/\mathbb{Z}_2 = SO(3)$ computed by A-C-M, in spite of the fact that the orbit is supersymmetric.

~~Twisted SU(2). Odd Wilson surface~~ Wierd super Lie algebras

$\mathfrak{g} = \mathfrak{p}_0 \oplus \Pi \mathfrak{h}_0$, where $\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{h}_0 = su(2)$



$su(2) = u(1) \oplus \mathfrak{p}_0$

$\omega\, su(2) = \Pi u(1) \oplus \mathfrak{p}_0$

$\mathcal{O}_\lambda \simeq S^2$ foliate $\mathfrak{g}^* \simeq \mathbb{R}^3$

$WS = e^{i\varphi} \longrightarrow 1 + \lambda$

Spin-off: Weird Lie Algebras

with Alexei Kotov
Vladimir Salnikov

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathfrak{h}$: subalgebra

$\mathfrak{m}$: $\mathfrak{h}$-module

$$[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h} \qquad [\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$$

$$[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$$

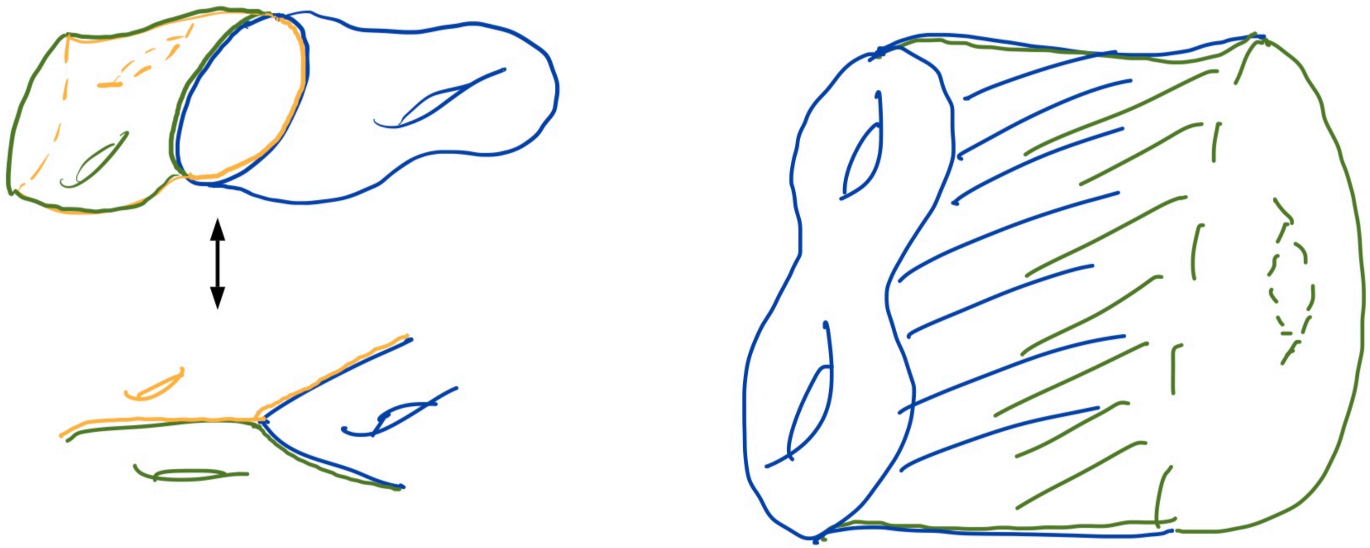
$$w\mathfrak{g} = \Pi\mathfrak{h} \oplus \mathfrak{m}$$

$[\ ,\ ]$ - odd

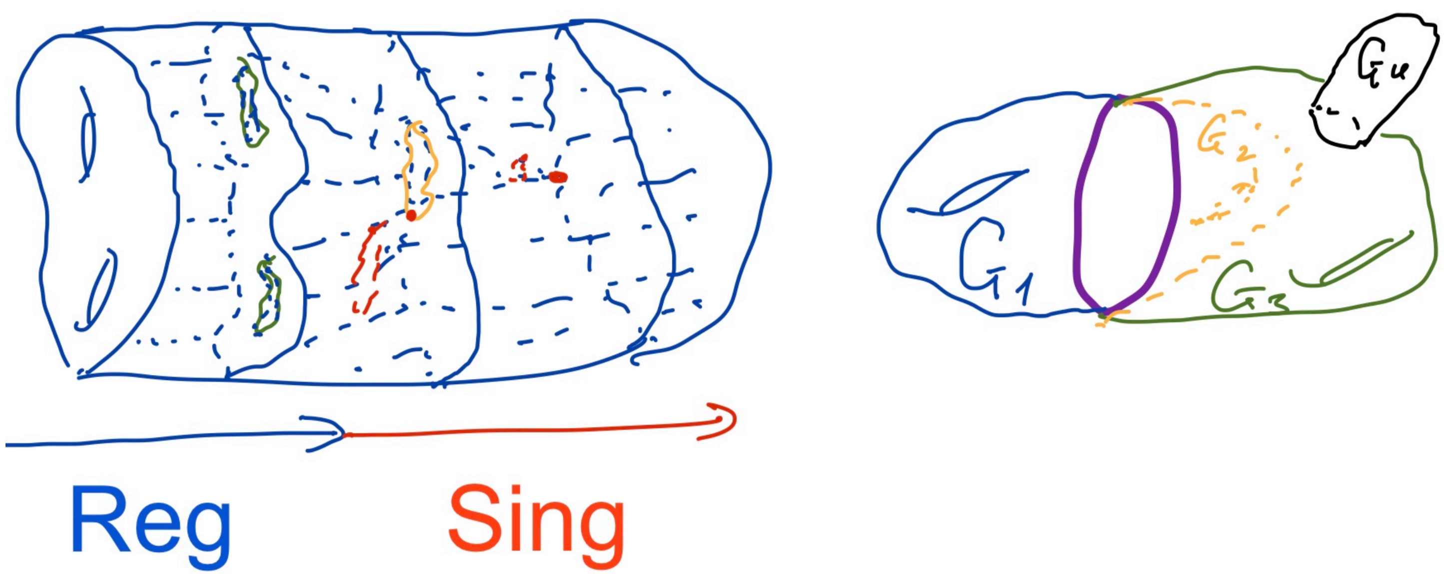
$$s\mathfrak{h} = \mathfrak{h} \oplus \Pi\mathfrak{m}, \quad [\ ,\ ] \text{ even}$$

$$(H, \mathfrak{h} \oplus \Pi\mathfrak{m}) \longleftarrow s\mathfrak{h}$$

Generalisations of Wilson surfaces Wilson surfaces en 2d



Close range and distant interactions



Continuous interactions and multiverse systems

Based on

[1] O. Chekeres, **Quantum Wilson surfaces and topological interactions**, Journal of High Energy Physics 02(2019)030

[2] A. Alekseev, O. Chekeres, P. Mnev, **Wilson surface observables from equivariant cohomology**, Journal of High Energy Physics 11(2015)093

[3] O. Chekeres, V. Salnikov, **Odd Wilson surfaces**, Journal of Geometry and Physics, 203 (2024) 105272

[...] Work in progress

Thank you

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