Iterated integrals and controlled ODEs

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Differential equations as models of physical systems

$x \equiv x(t)$ ... scalar quantity (state of a physical system at time t)

$$x(t+h) - x(t) = f(x(t))h$$
$$\downarrow$$
$$\dot{x}(t) = f(x(t))$$

Introducing noise - first attempt

$x \equiv x(t)$ ... scalar quantity (state of a physical system)

$\eta \equiv \eta(t)$ ... noise (random, non-systematic error)

$$y(t+h) - y(t) = f(y(t))h + g(y(t))\eta(t)h$$
$$\downarrow$$
$$\dot{y}(t) = f(y(t)) + g(y(t))\eta(t)$$

What properties should η have?

(i) $\eta(0) = 0$ (no error at the beginning)
(ii) independence (prior error does not influence future error)
(iii) stationarity (probabilistic properties do not change in time)
(iv) $\mathbb{E}\,\eta(1) = 0$, $\mathbb{E}\,\eta(1)^2 = 1$ (non-degeneracy)
(v) continuous paths

Problem: Such process does not exist!

Introducing noise - second attempt

$x \equiv y(t)$	...	scalar quantity (state of a physical system)
$W \equiv W(t)$	...	noise (random, non-systematic error)

$$y(t+h) - y(t) = f(y(t))h + g(y(t))[W(t+h) - W(t)]$$
$$\downarrow$$
$$\dot{y}(t) = f(y(t)) + g(y(t))\dot{W}(t)$$

What properties should W have?

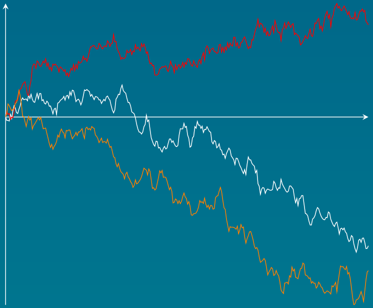
(i) $W(0) = 0$
(ii) independence of *increments*
(iii) stationarity of *increments*
(iv) $\mathbb{E}\, W(1) = 0,\ \mathbb{E}\, W(1)^2 = 1$
(v) continuous paths

The process W does exists. Let's call it the Brownian motion.

Differential equations with noise

$$\dot{y}(t) = f(y(t)) + g(y(t))\dot{W}(t)$$

Typical Brownian paths:



Problem: Brownian paths are nowhere differentiable.

Recall

$$y(t+h) - y(t) = f(y(t))h + g(y(t))[W(t+h) - W(t)]$$

and consider equidistant partition of $[0, T]$ with mesh smaller than h:

$$0 = \underbrace{t_0 \qquad\qquad t_1}_{= \Delta t \leq h} \qquad \cdots \qquad t_{n-1} \qquad t_n = T$$

We then have

$$y(T) - y(0) \quad = \quad \underbrace{\sum_{i=0}^{n-1} f(y(t_i))\Delta t_{i+1}}_{\substack{\downarrow \\ \int_0^T f(y(s))\,\mathrm{d}s}} \quad + \underbrace{\sum_{i=0}^{n-1} g(y(t_i))\Delta W_{t_{i+1}}}_{\substack{\downarrow \\ \int_0^T g(y(s))\,\mathrm{d}W_s}}$$

Question: Can the ODE be given meaning in the integral form?

What can we learn from the exponential?

Let us see what the bare minimum that we need is. Consider

$$\dot{y}(t) = y(t), \quad y(0) = y_0.$$

With $w(s) := s$, we can equivalently write

$$y(t) = y_0 + \int_0^t y(s)\,\mathrm{d}w(s).$$

Solution is found as the limit of Picard's iterations:

$$\begin{aligned} y^0(t) &= y_0, \\ y^n(t) &= y_0 + \int_0^t y^{n-1}(s)\,\mathrm{d}w(s), \quad n \in \mathbb{N}. \end{aligned}$$

The first few:

$$\begin{aligned} y^0(t) &= y_0, \\ y^1(t) &= y_0 \left[1 + \int_0^t 1 \, \mathrm{d}w(r)\right], \\ y^2(t) &= y_0 \left[1 + \int_0^t 1 \, \mathrm{d}w(r) + \int_0^t w(r) \, \mathrm{d}w(r)\right]. \end{aligned}$$

We see that we get Taylor's expansion of the exponential:

$$\begin{aligned} y^0(t) &= y_0, \\ y^1(t) &= y_0 \left(1 + t\right), \\ y^2(t) &= y_0 \left(1 + t + \frac{t^2}{2}\right). \end{aligned}$$

Continuity of the solution map

Observation: We need at least iterated integrals $\int_0^t W(r)\,\mathrm{d}W(r)$.

Problem: There is no continuous extension of the Stieltjes integral that could be used to define the iterated integral for a Wiener path.

This means that while for the discrete equation

$$y(T) - y(0) = \sum_{i=0}^{n-1} f(y(t_i))\Delta t_{i+1} + \sum_{i=0}^{n-1} g(y(t_i))\Delta W_{t_{i+1}},$$

the solution map

$$S : (\Delta W_{t_1}, ..., \Delta W_{t_n}; x(0)) \mapsto (y(t_1), ..., y(t_n))$$

is continuous, continuity is in general lost in the limit as $|\Delta t| \to 0$.

Itô vs. Stratonovič SDEs

Idea: Approximation by Riemann-type sums

$$I_W(t^*) := \sum_{i=0}^{n-1} W(t_i^*)\Delta W_{t_{i+1}}, \quad t_i^* \in [t_i, t_{i+1}]$$

is still desirable so change the mode of convergence - instead of almost sure convergence, consider convergence in probability.

But:

- By choosing different t^*, we obtain different objects:

$$\begin{array}{llll} t_i^* = t_i & \text{(Itô)} & \dots & \mathbb{E}\, I_W(t^*) = 0 \\ t_i^* = \frac{t_{i+1}+t_i}{2} & \text{(Strat)} & \dots & \mathbb{E}\, I_W(t^*) = \frac{t}{2} \\ t_i^* = t_{i+1} & & \dots & \mathbb{E}\, I_W(t^*) = t \end{array}$$

- We cannot hope to solve equations pathwise:

$$y(t) - y(0) = \int_0^t f(s, y(s))\,\mathrm{d}s + \text{Itô/Strat} \int_0^t g(s, y(s))\mathrm{d}W_s$$

A key observation

Well-posedness of the problem is restored if one has not only the sample path W but also its iterated integral $\int W \, \mathrm{d}W$ as input.

The Itô-Lyons solution map

The Itô solution map can be factorized as

$$W(\omega) \quad \overset{\Psi}{\longmapsto} \quad (W, \mathbb{W})(\omega) \quad \overset{\hat{S}}{\longmapsto} \quad y(\omega)$$

where

- Ψ is a measurable map that does not depend on $y(0)$, f, or g but only consists of enhancing the Wiener path with iterated integrals

$$\mathbb{W}_{s,t} = \int_s^t (W(r) - W(s))\,\mathrm{d}W(r)$$

- $\hat{S}$ is a (continuous!) map that takes enhanced path $(X, \mathbb{X})$ defined via certain algebraic properties and analytical conditions as input and spits out a solution of the (rough) differential equation

$$y(t) = y(0) + \int_0^t f(y(s))\,\mathrm{d}s + \int_0^t g(y(s))\,d(X, \mathbb{X})_s.$$

Construction of $\hat{S}$

Recall that we need to give meaning to the integral

$$\int_0^t g(y(s))\,\mathrm{d}W_s$$

where y is the (so far unknown) solution to the differential equation.

But here, x is not arbitrary! Solution to a differential equation should behave, at least locally, as the driver W, i.e.

$$y(t) - y(s) = g(y(s))[W(t) - W(s)] + R_{s,t}$$

where R is some remainder that is "smoother" than W.

But that means that we need to be able to define integrals of W against W itself and that is precisely what $\mathbb{W}$ encodes.

Rough path - Chen's relation

We wish to think of $\mathbb{X}_{s,t}$ as the iterated integral $\int_s^t (X_r - X_s)\,\mathrm{d}X_r$.

If we have a form of integration such that

- $f \mapsto \int f \mathrm{d}X$ is linear,
- $\int_s^t \mathrm{d}X_r = X_t - X_s$,
- $\int_s^t f \mathrm{d}X = \int_s^u f \mathrm{d}X + \int_u^t f \mathrm{d}X$ $(s \leq u \leq t)$,

then

$$\int_s^t (X_r - X_s)\mathrm{d}X_r - \int_s^u (X_r - X_s)\mathrm{d}X_r - \int_u^t (X_r - X_u)\mathrm{d}X_r = (X_t - X_u)(X_u - X_s)$$

or, in other words, $\mathbb{X}$ should satisfy

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_t - X_u)(X_u - X_s).$$

Rough path - regularity

If $X \in C^\alpha$, then (formally!),

$$\left|\int_s^t (X_r - X_s)\mathrm{d}X_r\right| \lesssim \int_s^t |r-s|^\alpha |\mathrm{d}X_r| \lesssim |t-s|^\alpha \int_s^t |\mathrm{d}X_r| \lesssim |t-s|^{2\alpha}$$

so that one would expect that $\mathbb{X} \in C^{2\alpha}$.

Rough path - definition

Definition. For $\alpha \in (1/3, 1/2]$, an α-Hölder rough path $\boldsymbol{X} = (X, \mathbb{X})$ is a pair of functions $X : [0, T] \to \mathbb{R}$ and $\mathbb{X} : [0, T]^2 \to \mathbb{R}$ such that

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_t - X_u)(X_u - X_s)$$

holds for every $s \leq u \leq t$ and such that

$$\|X\|_\alpha := \sup_{s \neq t} \frac{|X_t - X_s|}{|t - s|^\alpha} < \infty \quad \text{and} \quad \|\mathbb{X}\|_{2\alpha} := \sup_{s \neq t} \frac{|\mathbb{X}_{s,t}|}{|t - s|^{2\alpha}} < \infty.$$

Rough paths for the Wiener process

Let W be the Wiener process. We set

$$\mathbb{W}^{\text{Itô}}_{s,t} := (\text{Itô}) \int_s^t (W_r - W_s) \mathrm{d} W_r$$

$$\mathbb{W}^{\text{Strat}}_{s,t} := (\text{Strat}) \int_s^t (W_r - W_s) \mathrm{d} W_r.$$

Then for any $\alpha \in (1/3, 1/2)$, both

$$(W, \mathbb{W}^{\text{Itô}}_{s,t}) \quad \text{and} \quad (W, \mathbb{W}^{\text{Strat}}_{s,t})$$

are, almost surely, α-Hölder rough paths.

Rough integral - idea

Recall that we need to define

$$\int_0^{\bullet} Y_r \mathrm{d}\boldsymbol{X}_r$$

for Y that "looks like X on very small scales".

If X is nice enough (i.e. if $\alpha > 1/2$), we have that

$$Y_r \approx Y_s \cdot 1 \quad \text{then} \quad \int_0^1 Y_r \mathrm{d}X_r \approx \sum_{[s,t]\in\mathcal{P}} Y_s(X_t - X_s).$$

For rougher X (i.e. if $1/3 < \alpha \leq 1/2$), we should have

$$Y_r \approx Y_s \cdot 1 + Y'_s(X_r - X_s) \quad \text{then} \quad \int_0^1 Y_r \mathrm{d}\boldsymbol{X}_r \approx \sum_{[s,t]\in\mathcal{P}} Y_s(X_t - X_s) + Y'_s \mathbb{X}_{s,t}$$

Rough integral - controlled paths

Definition. Given $X \in C^\alpha$, we say that $Y \in C^\alpha$ is *controlled by X* if there exists $Y' \in C^\alpha$ such that the remainder R^Y defined by

$$Y_t - Y_s = Y'_s(X_t - X_s) + R^Y_{s,t}$$

satisfies $\|R^Y\|_{2\alpha} < \infty$.

Rough integral

Theorem. Let $\alpha \in (1/3, 1/2]$ and let $\boldsymbol{X} = (X, \mathbb{X})$ be an α-Hölder rough path. Let Y be a path controlled by X. Then the integral

$$\int_0^1 Y_s \mathrm{d}\boldsymbol{X}_s := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \left(Y_s(X_t - X_s) + Y'_s \mathbb{X}_{s,t}\right)$$

exists and for every pair $s \leq t$, we have

$$\left| \int_s^t Y_s \mathrm{d}\boldsymbol{X}_s - Y_s(X_t - X_s) - Y'_s \mathbb{X}_{t,s} \right| \lesssim_\alpha \left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha \right) |t - s|^{3\alpha}.$$

Rough integrals for the Wiener process

Recall that, for almost every ω, both

$$\boldsymbol{W}^{\text{Itô}} = (W(\omega), \mathbb{W}^{\text{Itô}}_{s,t}(\omega)) \quad \text{and} \quad \boldsymbol{W}^{\text{Strat}} = (W(\omega), \mathbb{W}^{\text{Strat}}_{s,t}(\omega))$$

are α-Hölder rough paths for any $\alpha \in (1/3, 1/2)$.

Theorem. Assume that $Y(\omega)$ is a path controlled by $W(\omega)$. Then both rough integrals

$$\int_0^1 Y_s(\omega) \mathrm{d}\boldsymbol{W}^{\text{Itô}}_s(\omega) \quad \text{and} \quad \int_0^1 Y_s(\omega) \mathrm{d}\boldsymbol{W}^{\text{Strat}}_s(\omega)$$

exist. If, moreover, Y is adapted, then, almost surely, these integrals coincide with

$$(\text{Itô}) \int_0^1 Y_s \mathrm{d}W_s \quad \text{and} \quad (\text{Strat}) \int_0^1 Y_s \mathrm{d}W_s,$$

respectively.

Rough differential equations

Problem: For given g, y_0, and a rough path $\boldsymbol{X} = (X, \mathbb{X})$, find y such that

$$y(t) = y(0) + \int_0^t g(y(s))\mathrm{d}\boldsymbol{X}_s.$$

Theorem. Let $\alpha \in (1/3, 1/2)$. Given $y(0) \in \mathbb{R}$, $g \in C^3$, and an α-Hölder rough path $\boldsymbol{X} = (X, \mathbb{X})$, there exists $T_0 \in (0, 1]$ and a unique path Y that is controlled by X with $Y' = g(Y_s)$, such that

$$Y_t = y_0 + \int_0^t g(Y_s)\mathrm{d}\boldsymbol{X}_s, \quad t \in [0, T_0].$$

(Proof idea: Banach fixed point in the space of controlled paths.)

Rough differential equations for the Wiener process

Recall that, for almost every ω, both

$$\boldsymbol{W}^{\text{Itô}} = (W(\omega), \mathbb{W}^{\text{Itô}}_{s,t}(\omega)) \quad \text{and} \quad \boldsymbol{W}^{\text{Strat}} = (W(\omega), \mathbb{W}^{\text{Strat}}_{s,t}(\omega))$$

are α-Hölder rough paths for any $\alpha \in (1/3, 1/2)$.

Theorem. Let $\alpha \in (1/3, 1/2)$. Given $y(0) \in \mathbb{R}$ and $g \in C^3_b$ for almost every ω, there is a unique solution (i.e. a path $Y(\omega)$ controlled by $W(\omega)$) to the RDE

$$Y_t(\omega) = y(0) + \int_0^t g(Y_s(\omega))\mathrm{d}\boldsymbol{W}^{\text{Itô/Strat}}(\omega)$$

that, almost surely, coincides with the solution to the SDE

$$Y_t = y(0) + (\text{Itô/Strat}) \int_0^t g(Y_s)\mathrm{d}W_s.$$

Wrap-up

- Let $\alpha \in (1/3, 1/2]$ and $g \in C_b^3$. Let X be an α-Hölder trajectory of a stochastic process. To solve

$$\mathrm{d}Y_t = g(Y_t)\mathrm{d}X_t, \quad Y_0 = y_0,$$

  we proceed in two steps:
  (1) Probabilistic step: Lift X to $(X, \mathbb{X})$.
  (2) Analytical: Solve the (R)DE pathwise.
- Key observation: Adding iterated integrals as *input* restores well-posedness of the problem. The solution map is continuous as a function of the (rough) path and the initial condition.
- Possible extensions for $\alpha \leq 1/3$ - add higher order terms that play the role of higher-order iterated integrals.
- For the rough path theory and for stochastic analysis, it is crucial to understand the analytical and algebraic properties of iterated integrals.

Thank you for your attention!

Some references:

(1) Friz, P.K., Hairer, M., A Course on Rough Paths: With an Introduction to Regularity Structures, Springer, 2014.

(2) Friz, P.K., Victoir, N.B., Multidimensional Stochastic Processes as Rough Paths: Theory and Applications, Cambridge University Press, 2010.