Globalizations of l_∞-algebras associated to dg-manifolds

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Outline

Q-manifolds

Definition (Q-manifold)

A Q-manifold is a graded manifold $\mathcal{M}$ together with a vector field Q of degree one such that $Q^2 = 0$.

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Examples

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- *Let $\mathfrak{g}$ be a Lie algebra. The shifted vector space $\mathfrak{g}[1]$ is a Q-manifold endowed with the Chevalley-Eilenberg differential.*

L_∞-algebras

Definition (L_∞-algebra)

Let L be a graded vector space. We say that L is an l_∞-algebra if it is endowed with graded symmetric brackets $l_k : L \times \cdots \times L \longrightarrow L$ of degree one satisfying:

$$\sum_{i+j=n+1} \sum_{\sigma \in \mathfrak{S}_{i,n-i}} \varepsilon(\sigma) l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma_(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0$$

where σ denotes a permutation such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$.

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3) For homogeneous $v_1, v_2, v_3 \in L$:

$$\begin{aligned} l_2(l_2(v_1, v_2), v_3) \pm l_2(l_2(v_1, v_3), v_2) \pm l_2(l_2(v_2, v_3), v_1) \\ + l_1(l_3(v_1, v_2, v_3)) + l_3(l_1(v_1), v_2, v_3) \\ \pm l_3(v_1, l_1(v_2), v_3) \pm l_3(v_1, v_2, l_1(v_3)) = 0, \end{aligned}$$

which means that the Jacobi identity holds in the cohomology of L.

L_∞-algebras as formal pointed Q-manifolds

A vector space endowed with an L_∞- structure can be seen as a pointed Q-manifold by defining the formal homological vector field Q as:

$$Q(v) = \sum_{i \geq 1} \frac{1}{i!} l_i(v, \dots, v).$$

for $v \in L$. It can be seen that Q is homological if and only if the brackets $\{l_i\}_{i \geq 1}$ define an L_∞-algebra.

Now, consider a general Q-manifold $\mathcal{M}$ and let $p \in M$. The homological vector field has a Taylor expansion:

$$Q^k(x) = Q^k(p) + Q^k_i(p)x^i + \frac{1}{2}Q^k_{ij}(p)x^i x^j + \dots$$

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$$l_i : T_p\mathcal{M}^{\otimes i} \longrightarrow T_p\mathcal{M}, \qquad l_i(e_{b_1}, \dots, e_{b_i}) := Q^k_{b_1 \dots b_i} e_k$$

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In general, if $Q^k(p) \neq 0$, such term is known as the curvature and the corresponding structure is called "curved" l_∞-algebra.

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What do we wish? We would like to introduce a construction that is independent of local coordinates.

Fedosov-like construction for graded manifolds

Consider the bundle whose fiber at a point $x \in M$ is $\hat{S}(T^*_x\mathcal{M}) \otimes T_x\mathcal{M}$.
If $\{y^i\}_{i=1}^n$ and $\{p_j\}_{j=1}^n$ denote coordinates on $T_x\mathcal{M}$ and $T^*_x\mathcal{M}$ respectively, then we can write an element in the fiber $a_x \in \hat{S}(T^*_x\mathcal{M}) \otimes T_x\mathcal{M}$ as follows:

$$a_x = \sum_{i=1}^n a^i(y) p_i$$

where $a^i(y)$ is a formal power series on y^i's variables.

Lie bracket

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Definition

Let $\{y^i\}$ and $\{p_j\}$ be coordinates in $T_x\mathcal{M}$ and $T_x^*\mathcal{M}$ respectively, then we can express the (fiberwise) Lie bracket between formal vector fields as:

$$[a_x, b_x] = \sum_{i=1}^{n} \left(a^i \frac{\partial}{\partial y^i} b^j - (-1)^{|a||b|} b^i \frac{\partial}{\partial y^i} a^j \right) p_j$$

for $a_x = \sum_{i=1}^n a^i(y) p_i$, $b_x = \sum_{i=1}^n b^i(y) p_i \in \hat{S}(T_x^*\mathcal{M}) \otimes T_x\mathcal{M}$.

Derived brackets

Proposition

Let L be a Lie superalgebra and $\Delta \in L$ a nilpotent odd element. If there exists a projector P from L onto an abelian subalgebra $\mathcal{A}$ satisfying the distributivity condition

$$P[a, b] = P[Pa, b] + P[a, Pb], \ \ \forall a, b \in L$$

then there exists an $l_\infty-$algebra on $\mathcal{A}$ with the n-ary bracket given by:

$$\lambda_n(a_1, \dots, a_n) = P[\dots[[\Delta, a_1], a_2], \dots], \ \ \forall a_1, \dots, a_n \in \mathcal{A}.$$

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So, in order to define an $l_\infty-$ algebra on the space of vector fields $\Gamma(T\mathcal{M})$, we need to construct a nilpotent operator $\Delta \in \Gamma(\hat{S}(T^*\mathcal{M}) \otimes T\mathcal{M})$ and a projector P from $\Gamma(\hat{S}(T^*\mathcal{M}) \otimes T\mathcal{M})$ to the space of vector fields $\Gamma(T\mathcal{M})$.

The key results

Theorem

Let ∇ be a torsion-free connection on $T\mathcal{M}$. Then there exists a one-form $\mathbf{r}$ with values in $\hat{S}(T^*\mathcal{M}) \otimes T\mathcal{M}$ such that the operator $D := \nabla + [\mathbf{r}, \cdot]$ is nilpotent.

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Notation. Given a vector field $X \in \Gamma(T\mathcal{M})$, we will denote the corresponding covariantly constant element in $\Gamma(\hat{S}(T^*\mathcal{M}) \otimes T\mathcal{M})$ by $\hat{X}$.

By applying the previous lemmas to the homological vector field Q, we obtain a nilpotent and covariantly constant element $\hat{Q} \in \Gamma(\hat{S}(T\mathcal{M}) \otimes T\mathcal{M})$ such that $\hat{Q}\big|_{y=0} = Q$. Then, by the previous proposition we obtain the following brackets:

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The unary map $\lambda_1 : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T\mathcal{M})$ is defined by: $\lambda_1(X) = [\hat{Q}, X]\big|_{y=0}$, for $X \in \Gamma(T\mathcal{M})$. This will correspond to $-(-1)^{|X|}\nabla_X Q$.

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- The binary map $\lambda_2 : \Gamma(T\mathcal{M}) \otimes \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T\mathcal{M})$ is given by $\lambda_2(X, Y) = [[\hat{Q}, X], Y]\big|_{y=0}$.

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In general, for the n-ary bracket:

$$\lambda_n(X_1, \ldots, X_n) = [\ldots[[\hat{Q}, X_1], X_2], \ldots X_n]\big|_{y=0},$$

where $X_1, \ldots, X_n \in \Gamma(T\mathcal{M})$.

Kapranov-type l_∞- algebras

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The binary bracket $\lambda_2 : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T\mathcal{M})$ coincides with the so-called Atiyah cocycle:

$$\lambda_2(X, Y) = [Q, \nabla_X Y] - \nabla_{[Q,X]} Y - (-1)^{|X|} \nabla_X [Q, Y]$$

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$$\lambda_2(X, Y) = [Q, \nabla_X Y] - \nabla_{[Q,X]} Y - (-1)^{|X|} \nabla_X [Q, Y]$$

In general, they showed that λ_n, for $n \geq 2$ is recursively determined by the Atiyah cocycle and the curvature of the connection.

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For a torsion-free connection, we obtain:

$$\lambda_1(X) = [Q, X] = \nabla_Q X - (-1)^{|X|}\nabla_X Q, \ \forall X \in \Gamma(T\mathcal{M})$$

for the Kapranov l_∞-algebra, while

$$\lambda_1(X) = -(-1)^{|X|}\nabla_X Q$$

for the Fedosov procedure.

We wish to have a general understanding on how to obtain both structures from a more general and unifying approach.

Geodesic exponential map

Let $p_1, p_2 : M \times M \longrightarrow M$ be the two projections onto M. By considering the geodesic exponential map

$$\exp : TM \longrightarrow M \times M$$

we identify the formal neighborhood of the diagonal $M^{(\infty)}$ with the formal neighborhood of the zero section of TM, denoted by Δ_0^∞. We can reproduce the two previous l_∞- algebras by considering two different liftings of the homological vector field Q.

- We can lift Q by the condition $dp_i(\hat{Q}) = Q$ for $i = 1, 2$, or
- We can consider $dp_1(\hat{Q}) = 0$ and $dp_2(\hat{Q}) = Q$.

Flat coordinates

If $(x^1, \ldots, x^n, \eta^1, \ldots \eta^n)$ denotes coordinates on $T\mathcal{M}$, the geodesic coordinates on $\mathcal{M} \times \mathcal{M}$ take the form $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, where $y^i = x^i + \eta^i$.

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First lifting We consider the lifting of Q given by: $\hat{Q}(x, y) = Q^i(x)\frac{\partial}{\partial x^i} + Q^j(y)\frac{\partial}{\partial y^j}$.
We express $\hat{Q}$ in coordinates (x, η) and expand around $\eta = 0$. By applying the construction of derived brackets (Voronov [27]), we obtain an l_∞- algebra in the space of vector fields of the form $X = X^i(x)\frac{\partial}{\partial \eta^i}$ whose brackets coincide (up to some factors) with the Kapranov l_∞- algebra.

Flat case

Second lifting On the other hand, if we consider the lifting given by $\hat{Q}(x,y) = Q^i(y)\frac{\partial}{\partial y^i}$ and proceed as in the previous way, the construction of derived brackets produces an l_∞-algebra that coincides with the one obtained in the Fedosov-like procedure.

Non-zero curvature

In general, we also expect to reproduce the two l_∞-algebras in the case where we have general geodesic coordinates.

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The next steps

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The next steps

- We write the geodesic coordinates (x, y) in terms of coordinates (x, η) on TM by a formal power series in η.
- Next, we reconstruct the Fedosov operator D. This will coincide with the one mentioned previously.
- We proceed to construct the lifting of the homological vector field Q, written in geodesic coordinates, and compute the corresponding l_∞-algebra. We plan to prove that this coincides with the Fedosov l_∞-algebra.

Děkuju!

¡Gracias!

Thank you!

Danke!

Grazie!

Merci!

Dziękuję!

Hvala ti!

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