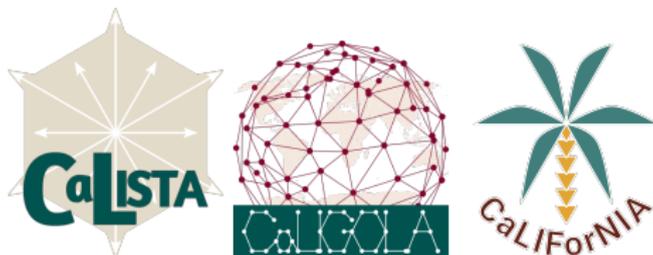


# Geometric Deep Learning meets Quantum Groups

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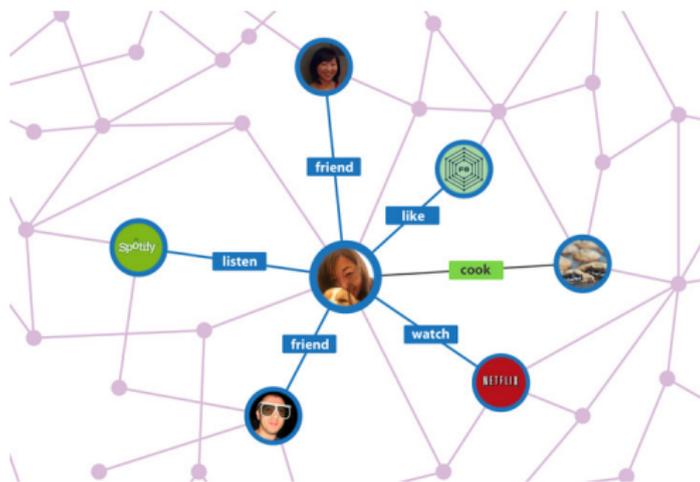
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- Machine Learning, Geometric Deep Learning need a theory of differential operators on graphs (meshes).
- Quantum Geometry shows that quantum differential calculus is the right framework to write geometry on graphs.
- Sheaf Neural Networks show greater “expressibility” because of the use of rings of functions versus just function values.

# Directed Graphs as Semisimplicial sets

- $\Delta_+$ : category with objects the ordered sets  $[n] = \{0 < \dots < n\}$ ,  $n \in \mathbb{N}$ , and arrows the injective order preserving maps between them.
- $\Delta_{n,+}$ : category with objects the ordered sets  $[n] = \{0 < \dots < n\}$ ,  $n$  fixed.
- $\text{ssSets} := \text{Fun}(\Delta_+^{\text{op}}, \text{Sets})$ : category of semisimplicial sets.

## Example

$\text{diGraphs} = \text{Fun}(\Delta_{1,+}^{\text{op}}, \text{Sets})$ ,  $\Delta_{1,+}$  has objects  $[0] = \{0\}$  and  $[1] = \{0 < 1\}$ .

$$\begin{aligned} G : \quad \Delta_{1,+}^{\text{op}} &\longrightarrow \text{Sets} \\ [0] &\mapsto G([0]) = V_G, \\ [1] &\mapsto G([1]) = E_G, \\ [0] \hookrightarrow [1] &\mapsto h : E_G \rightarrow V_G, \quad t : E_G \rightarrow V_G \end{aligned}$$

$$\begin{array}{ccccc} & & E_G & & \\ & h_G \swarrow & \downarrow i & \searrow t_G & \\ V_G & \xleftarrow{\text{pr}_1} & V_G \times V_G & \xrightarrow{\text{pr}_2} & V_G \end{array}$$

(1)

**Attention:**  $i$  is not necessarily injective!

## Definition

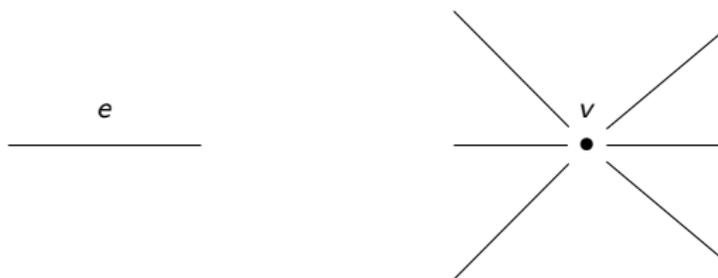
For  $G = (V, E)$  we define the poset (partially ordered set) structure:

$x \leq y$  if and only if  $x = y$  or  $x$  is a vertex of the edge  $y$ .

We define a topology generated by the base of open sets

- $U_v = \{e \in E \mid v \leq e\}$ , that is the open star of  $v$ , for each vertex  $v \in V$ ,
- $U_e = \{e\}$ , i.e. the edge  $e$ , without its vertices, for each  $e \in E$ .

**Irreducible open sets:**



**Irreducibles for the dual topology** (open are the closed subsets):



## Theorem (Key Result)

Let  $X$  be a topological space. If  $X$  has a basis consisting of irreducible open sets, then there is an equivalence between:

presheaves on irreducible open sets in  $X$   $\Leftrightarrow$  sheaves on  $X$ .

## Observation

A sheaf of vector spaces on a digraph  $G = (E_G, V_G, h_G, t_G)$  for the standard (dual) topology is equivalent to give

- a vector space  $F(v)$  for each vertex  $v \in V_G$ ,
- a vector space  $F(e)$  for each edge (with its endpoints)  $e \in E_G$ ,
- linear maps (restriction maps)  $F_{h_G(e) \leq e} : F(e) \rightarrow F(h_G(e))$ ,  
 $F_{t_G(e) \leq e} : F(e) \rightarrow F(t_G(e))$  for each edge  $e \in E_G$ , where, we write  $v \leq e$  to mean that  $v$  is a vertex of the edge  $e$ .

## Observation (Irreducible open sets in the dual standard topology)



This is the topology and the sheaf definition used in Geometric Deep Learning.

## Definition

Let  $G \in \text{diGraphs}$ . We say that the surjective map  $\phi : H \rightarrow G$  is an **étale directed cover** if

- 1  $H$  is a disjoint union of graphs in  $\text{diGraphs}_{\leq 1}$ .
- 2 The arrow  $\phi_E : E_H \rightarrow E_G$  induced by  $\phi$  is bijective when restricted to non self-loops.

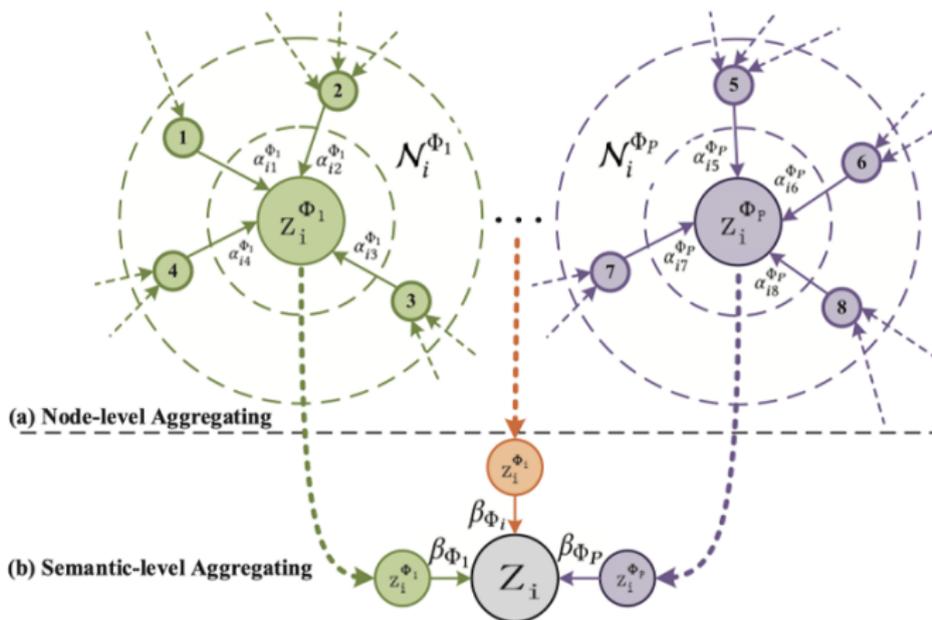


Clearly, given  $G$ , such  $H$  and  $\phi$  are not unique, but they always exist.

## Remark

It is possible to define Grothendieck topologies and étale coverings more general on semisimplicial sets together with their sheaves (sites and topos).

## Heterogeneous Graph Attention Network



Ref: "Heterogeneous Graph Attention Network",  
<https://arxiv.org/pdf/1903.07293>

## Definition

A **first order differential calculus (FODC)** on an algebra  $A$  is  $(\Gamma, d)$ , where

- i.)  $\Gamma$  is an  $A$ -bimodule.
- ii.)  $d: A \rightarrow \Gamma$  is a  $\mathbf{k}$ -linear map satisfying the Leibniz rule

$$d(ab) = d(a)b + ad(b)$$

for all  $a, b \in A$ .

- iii.)  $A \otimes A \rightarrow \Gamma$ ,  $a^i \otimes b^j \mapsto a^i d(b^j)$  is a (left  $A$ -linear and) surjective map.

## Example (Kähler differential, exterior derivative)

Take  $A = C^\infty(M)$ ,  $M$  differentiable manifold,  $\Gamma = \Omega^1(M)$ .

$$d: C^\infty(M) \longrightarrow \Omega^1(M), \quad f \mapsto df$$

In local coordinates:

$$df = \sum \partial_i f^i dx_i$$

# First Order Differential Calculus on diGraphs

Let  $G = (V, E) \in \text{diGraphs}_{\leq 1}$  ( $G$  directed with at most one edge per direction).

$$A := \mathbf{k}[V] = \text{span}\{\delta_x \mid x \in V\}, \quad (2)$$

where  $\delta_x(y) = 1$  if  $x = y$  and zero otherwise.

## Definition

We define a FODC  $(\Gamma^1, d)$ , on  $A = \mathbf{k}[V]$

$$\Gamma^1 := \mathbf{k}[E] = \text{span}\{\omega_{x \rightarrow y} \mid (x, y) \in E\}$$

The  $A$ -bimodule structure is given by:

$$f\omega_{x \rightarrow y} = f(x)\omega_{x \rightarrow y}, \quad \omega_{x \rightarrow y}f = \omega_{x \rightarrow y}f(y), \quad df = \sum_{x \rightarrow y \in E} (f(y) - f(x))\omega_{x \rightarrow y}$$

We define  $d : A \rightarrow \Gamma^1$  on generators as:

$$d\delta_x = \sum_{y:y \rightarrow x} \omega_{y \rightarrow x} - \sum_{y:x \rightarrow y} \omega_{x \rightarrow y}, \quad \delta_x d\delta_y = \begin{cases} -\sum_{z:x \rightarrow z} \omega_{x \rightarrow z} & x = y \\ \omega_{x \rightarrow y} & x \rightarrow y \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

This FODC is **inner**, i.e.  $da = [\theta, a]$  for all  $a \in A$ , where

$$\theta := \sum_{x \rightarrow y \in E} \omega_{x \rightarrow y}$$

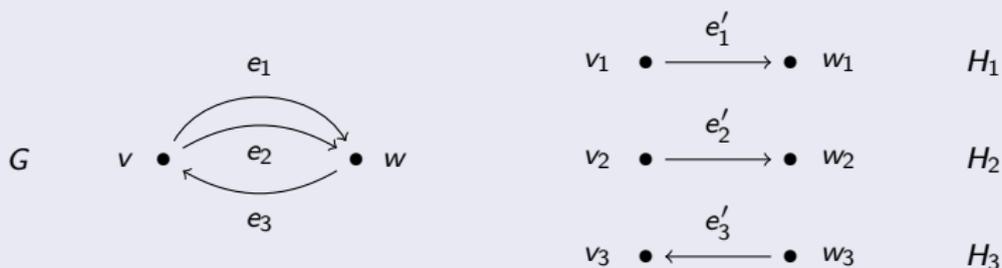
We can extend the theory of FODC to the case of multi-edge graphs.  
We illustrate it by an example.

## Example

Consider the graph  $G$  and its étale covering

$$f : H = H_1 \amalg H_2 \amalg H_3 \longrightarrow G$$

with  $G, H \in \text{diGraphs}$  (self-loops are not depicted)



We have that  $\Gamma_G^1 := \text{span}\{\omega_{v_1 \rightarrow w_1}, \omega_{v_2 \rightarrow w_2}, \omega_{v_3 \rightarrow w_3}\}$ , and

$$d(a) = d_1(f^*(a)|_{V_{H_1}}) + d_2(f^*(a)|_{V_{H_2}}) + d_3(f^*(a)|_{V_{H_3}}) \in \Gamma^1 = \Gamma_{H_1}^1 \oplus \Gamma_{H_2}^1 \oplus \Gamma_{H_3}^1$$

So, for example, if  $a = \delta_v$ ,  $d(\delta_v) = d_1(\delta_{v_1}) + d_2(\delta_{v_2}) + d_3(\delta_{v_3})$ .

## Definition (Braune et al. 2017)

A **vector bundle**  $\mathcal{F}$  of rank  $n$  on a set  $V$  is an assignment:

$$v \longrightarrow \mathcal{F}_v, \quad v \in V$$

where  $\mathcal{F}_v$  is a vector space of dimension  $n$ . We define the **frame bundle**  $\text{Fr}$ , an assignment:

$$V \ni v \mapsto \{e_i^v\} \subset \mathcal{F}_v$$

where  $\{e_i^v\}$  is a basis for  $\mathcal{F}_v$ . Moreover we denote with  $\mathbb{1}_{u,v} : \mathcal{F}_u \longrightarrow \mathcal{F}_v$  the linear map  $\mathbb{1}_{u,v}(e_i^u) = e_i^v$ .

## Definition

Let  $\mathcal{F}$  be a vector bundle on  $V$  and let  $G = (V, E) \in \text{diGraphs}$ .

- We define a **weak parallel transport** a collection of linear maps  $\mathcal{R}_{e,u \rightarrow v} : \mathcal{F}_v \longrightarrow \mathcal{F}_u$ , where  $e$  is an edge between  $u$  and  $v$ .
- If  $G \in \text{diGraphs}_{\leq 1}$  is bidirected, we say that a weak parallel transport is a **parallel transport** if each  $\mathcal{R}_{e,u \rightarrow v}$  is invertible and  $\mathcal{R}_{e,u \rightarrow v} = \mathcal{R}_{e',v \rightarrow u}^{-1}$ .

## Definition (Braune et al. 2017)

We define a **connection** on a digraph  $G$  as a collection of linear maps  $\Theta_{e,u \rightarrow v} := \mathcal{R}_{e,u \rightarrow v} - \mathbb{1}_{v,u}$ , on all edges  $e \in E$ , with  $\{\mathcal{R}_{e,u \rightarrow v}\}$  a weak parallel transport.

Once a frame bundle is given, we can write:

$$\mathcal{R}_{e,u \rightarrow v} : \mathcal{F}_v \longrightarrow \mathcal{F}_u, \quad e_i^v \mapsto \mathcal{R}_{e,u \rightarrow v,i}^j e_j^u$$

## Observation

In the differentiable setting the parallel transport for a vector bundle  $E \rightarrow M$  on a differentiable manifold  $M$  is a collection of maps:

$$\Gamma(\gamma)_s^t : E_{\gamma(s)} \rightarrow E_{\gamma(t)}$$

It allows us to take the derivative of a section  $V$  along a curve  $\gamma$ :

$$\nabla_{\dot{\gamma}} V = \lim_{h \rightarrow 0} \frac{\Gamma(\gamma)_h^0 V_{\gamma(h)} - V_{\gamma(0)}}{h} = \left. \frac{d}{dt} \Gamma(\gamma)_t^0 V_{\gamma(t)} \right|_{t=0}.$$

Rewrite replacing the curve  $\gamma$  with an edge  $e$  between vertices  $u$  and  $v$  (taking the places of  $\gamma(0)$  and  $\gamma(h)$ ) of the graph  $G \in \text{Graphs}$ .

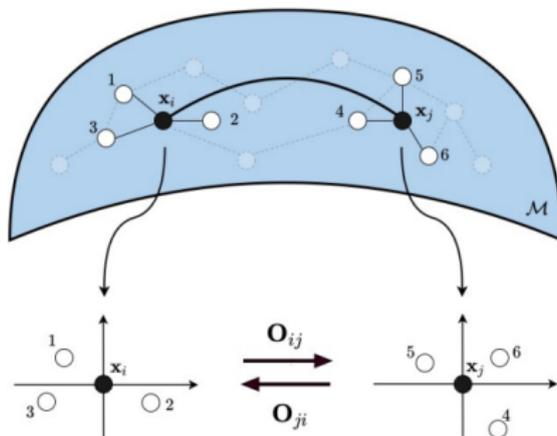
$$V_u \mapsto \mathcal{R}_{e,u \rightarrow v} V_v - V_u$$

## Observation

*Classically there is a correspondence:*

*Locally constant sheaves  $\leftrightarrow$  vector bundles with a flat connection*

*In machine learning the invertibility assumption on parallel transport  $\mathcal{R}_{x \rightarrow y} = \mathcal{R}_{y \rightarrow x}^{-1}$  severely restricts the connection, making it a flat one!*



Barbero et al. <https://arxiv.org/pdf/2206.08702>

## Definition

Let  $(\Gamma_G^1, d_G)$  be a FODC on  $\mathbf{k}[V]$  associated to  $G = (V, E) \in \text{diGraphs}$ . We define

- 1 a quantum metric on  $\Gamma^1$ , a bimodule map

$$(\cdot, \cdot) : \Gamma_G^1 \otimes_{A_G} \Gamma_G^1 \longrightarrow A_G$$

- 2 A  $\mathbf{k}$ -linear map  $\Delta : A_G \rightarrow A_G$  is a second order Laplacian if

$$\Delta(ab) = (\Delta a)b + a\Delta b + 2(da, db)$$

- 3 Graph laplacians associated to the metric  $(\cdot, \cdot)$  are given by:

$$\Delta_\theta(a) := 2(\theta, da) \quad , \quad \theta\Delta(a) := -2(da, \theta)$$

where  $\theta = \sum \omega_{x \rightarrow y}$ .

## Proposition

If we fix the basis  $\{\delta_x\}_{x \in V_G}$  for  $\mathbf{k}[V_G]$ , we identify  $\mathbf{k}[V_G] \cong \mathbf{k}^{|V_G|}$ ,  $L$  is a linear operator and one can readily check:

$$L = D - A = (1/2)\Delta_\theta, \quad \text{for } \lambda_{v \rightarrow w, w \rightarrow v} = \lambda_{w \rightarrow v, v \rightarrow w} = 1$$

where  $D$  is the degree matrix (diagonal matrix with the degree of vertices on the diagonal) and  $A$  is the adjacency matrix of  $G$ .

We can extend the definition  $\theta\Delta$  when a right connection is given on a vector bundle.

## Definition

Assume we have:

- $G \in \text{diGraphs}_{\leq 1}$  and a FODC  $\Gamma^1$ ,
- $M$  a free rank  $n$  right  $A_G$ -bimodule,
- $\nabla$  a right connection
- $(,)$  a generalized quantum metric on  $\Gamma_G^1$ .

Let  $\eta$  be the left  $A_G$ -module map  $M \otimes_A \Gamma_G^1 \rightarrow M \otimes_A \Gamma_G^1 \otimes_A \Gamma_G^1$ :

$$\eta(m \otimes \omega_{x \rightarrow y}) = m \otimes \omega_{x \rightarrow y} \otimes \theta$$

Define the **connection Laplacian**

$$\theta\Delta^M := -2(\mathbb{1} \otimes (,)) \circ \eta \circ \nabla : M \rightarrow M$$

## Observation

We have:

$$\theta\Delta^M(e_i f^i) = -2 \sum_{x \rightarrow y} \lambda_{x \rightarrow y \rightarrow x} (\mathcal{R}_{x \rightarrow y, i}^j f^j(y) - f^i(x)) e_j \delta_x$$

## Definition (Bodnar et al. 2022)

Let  $G = (V, E)$  be a directed graph,  $f \in \mathbf{k}[V]$ . Let  $\mathcal{F}$  be a sheaf of vector spaces. We define **sheaf Laplacian**

$$L_{\mathcal{F}}(f)_x := \sum_{y, x \leq x \rightarrow y} F_{x \leq x \rightarrow y}^{-1} (F_{x \leq x \rightarrow y} f_x - F_{y \leq x \rightarrow y} f_y) \quad (4)$$

Recall that a sheaf of vector spaces on a digraph  $G = (E_G, V_G, h_G, t_G)$  for the standard (dual) topology is equivalent to give

- a vector space  $F(v)$  for each vertex  $v \in V_G$ ,
- a vector space  $F(e)$  for each edge (with its endpoints)  $e \in E_G$ ,
- linear maps (restriction maps)  $F_{x \leq x \rightarrow y} : F(x \rightarrow y) \rightarrow F(x)$ ,  
 $F_{y \leq x \rightarrow y} : F(x \rightarrow y) \rightarrow F(y)$  for each edge  $x \rightarrow y \in E_G$ , where, we write  $v \leq e$  to mean that  $v$  is a vertex of the edge  $e$ .

where the irreducible open sets in the dual standard topology are



## Observation

*Vector bundles are locally free sheaves (as in ordinary geometry).*

## Theorem (F.-Simonetti-Zanchetta 2025)

*Assume*

- $G \in \text{diGraphs}_{\leq 1}$  is a bidirected graph with  $(\Gamma^1, d)$ , differential calculus
  - $\mathcal{F}$  a vector bundle i.e. a sheaf of vector spaces of rank  $n$  on  $G$
  - $\nabla$  connection with  $\mathcal{R}$  weak parallel transport,
  - $M$  the free right  $A_G$ -module associated to the vector bundle  $\mathcal{F}$ .
- 1 If  $\mathcal{R}^F$  is a parallel transport, then  ${}_{\theta}\Delta^M = -L_F$ .
  - 2 If  $\mathcal{F}$  is a sheaf of inner product spaces and  $F_{v \leq e}^* = F_{v \leq e}^{-1}$  (i.e.  $F$  in an  $O(n)$ -bundle), then

$$\nabla^* \nabla = L_F$$

where we fix isomorphisms  $M \cong M^*$  and  $\Gamma^1 \cong (\Gamma^1)^*$ .

## Remark

*This theorem can be proven more generally in the context of semisimplicial sets and their homology/cohomology.*

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Theorem (Dimakis 1994, Majid 2013)

We have a fully faithful contravariant functor

$$F : \text{diGraphs}_{\leq 1} \longrightarrow (\text{FODC}), \quad G = (V, E) \mapsto (\Gamma^1, d)$$

realizing an antiequivalence of categories between  $\text{diGraphs}_{\leq 1}$  and the category of FODC  $(\Gamma^1, d)$  on  $\mathbf{k}$ -algebras  $A = \mathbf{k}[V]$ , with  $V$  finite set.

We can extend the definition of FODC from  $\text{diGraphs}_{\leq 1}$  to  $\text{diGraphs}$ , obtaining still an equivalence of categories.

Theorem (F., Simonetti, Zanchetta 2025)

We have a fully faithful contravariant functor

$$\mathcal{F} : \text{diGraphs} \longrightarrow (\text{FODC})_e, \quad G \mapsto (\Gamma_G^1, d)$$

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## Definition

For a given  $G \in \text{diGraphs}_{\leq 1}$ , we define  $\Gamma_G^2$  as the vector space freely generated by the triangular cliques:

$$\Gamma_G^2 := \text{span}\{\omega_{x \rightarrow y \rightarrow z} \mid x \rightarrow y, y \rightarrow z \in V, x \neq y, y \neq z\} \quad (5)$$

$\Gamma_G^2$  is an  $A_G$ -bimodule:

$$f\omega_{x \rightarrow y \rightarrow z} = f(x)\omega_{x \rightarrow y \rightarrow z}, \quad \omega_{x \rightarrow y \rightarrow z}f = \omega_{x \rightarrow y \rightarrow z}f(z), \quad f \in A_G$$

In analogy to the continuous setting, we refer to  $\Gamma_G^2$  as the space of 2-forms on  $A_G$ .

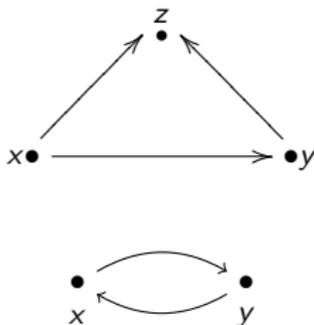


Figure: Triangular cliques

## Definition

Define  $\Omega_V^2$  to be the  $\mathbf{k}$ -vector space freely generated by the triangular cliques of fully connected  $G$  on  $V$ :

$$\Omega_V^2 := \text{span}\{\omega_{x \rightarrow y \rightarrow z} \mid x, y, z \in V, x \neq y, y \neq z\}$$

We define the **exterior product** as the  $\mathbf{k}$ -linear map:

$$\Omega_V^1 \times \Omega_V^1 \longrightarrow \Omega_V^2, \quad (\omega_{x \rightarrow y}, \omega_{w \rightarrow z}) \mapsto \omega_{x \rightarrow y} \wedge \omega_{w \rightarrow z} := \delta_{y,w} \omega_{x \rightarrow y \rightarrow z}$$

where  $(\Omega_V^1, d_V^0)$  is the FODC associated with the fully connected graph  $G$ .

$$d_V^1 : \Omega_V^1 \longrightarrow \Omega_V^2 \quad d_V^1 \omega_{x \rightarrow y} := d_V^0 \delta_x \wedge d_V^0 \delta_y \quad (6)$$

The map  $d_V^1$  satisfies the Leibniz rule:

$$d_V^1(f \omega_{x \rightarrow y}) = d_V^0 f \wedge \omega_{x \rightarrow y} + f d_V^1 \omega_{x \rightarrow y}, \quad d_V^1(\omega_{x \rightarrow y} f) = d_V^1 \omega_{x \rightarrow y} f - \omega_{x \rightarrow y} \wedge d_V^0 f$$

and  $d_V^1 \circ d_V^0 = 0$ . We also have the explicit expression:

$$d \omega_{x \rightarrow y} = \sum_{u \in V} (\omega_{u \rightarrow x \rightarrow y} - \omega_{x \rightarrow u \rightarrow y} + \omega_{x \rightarrow y \rightarrow u}) \quad (7)$$

Let  $G = (V, E) \in \text{diGraphs}_{\leq 1}$ , we can write the FODC  $\Gamma_G^1$  as a quotient of a **universal calculus**  $\Omega^1$  corresponding to the fully connected graph with vertices  $V$ :

$$\Gamma_G^1 = \Omega_V^1 / I, \quad I = \text{span}\{\omega_{x \rightarrow y} \mid x \rightarrow y \notin E\}$$

The previous proposition, along with the definition of the wedge product, allows us to see  $\Omega_V^\bullet := \bigoplus_{i=0}^2 \Omega_V^i$  (where  $\Omega_V^0 := \mathbf{k}[V]$ ) as a differential graded algebra  $(\Omega_V^\bullet, d_V^\bullet)$  (DGA)

## Proposition

Let  $G \in \text{diGraphs}_{\leq 1}$ . The graded  $A_G$ -bimodule  $\Gamma_G^\bullet := \bigoplus_{i=0}^2 \Gamma_G^i$ , where

$$\Gamma_G^0 := A_G, \quad \Gamma_G^1 = \Omega_V^1 / I \quad \Gamma_G^2 \cong \Omega_V^2 / d_V^1(I)$$

has a well defined DGA structure induced by the one of  $(\Omega_V^\bullet, d_V^\bullet)$ , the bimodule structure being the same.

We notice that any quotient of  $\Gamma_G^2$  by the span of a subset of the triangular cliques will give a well defined differential.

## Definition

Let  $G = (V, E) \in \text{diGraphs}_{\leq 1}$  and  $S$  a subset of its triangular cliques. We define the pair  $(\Gamma_S^\bullet, d_S^\bullet)$  with:

$$\Gamma_S^\bullet := \Gamma_G^\bullet / \langle S \rangle, \quad d_S^\bullet : \Gamma_S^\bullet \longrightarrow \Gamma_S^\bullet \quad (8)$$

a **second order differential calculus** on  $A = \mathbf{k}[V]$ , where  $\langle S \rangle$  is the  $A_G$ -bimodule generated by  $S$  and  $d_S^1$  is obtained from  $d_G^1$ , by taking the quotient of  $\Gamma_G^2$  by  $\langle S \rangle$ .

Note that if  $S = \emptyset$  we get that  $(\Gamma_S^\bullet, d_S^\bullet) = (\Gamma_G^\bullet, d_G^\bullet)$ . In addition, notice that  $d_S^1$  satisfies the Leibnitz rule and  $d_S^1 \circ d_S^0 = 0$ , where  $d_S^0 = d_G^0$  and  $\Gamma_S^i := \Gamma_G^i$  for  $i = 0, 1$ .

## Remark

Let  $V$  be a finite set. Our approach could be extended to obtain all differential graded algebras on  $A$  as quotients of the universal one  $\Omega_V := \bigoplus_n \Omega_V^n$ . Moreover one could also extend our results to comprehend the case of étale directed covers,

Theorem (Dimakis 1994, Majid 2013)

We have a fully faithful contravariant functor

$$F : \text{diGraphs}_{\leq 1} \longrightarrow (\text{FODC}), \quad G = (V, E) \mapsto (\Gamma^1, d)$$

realizing an antiequivalence of categories between  $\text{diGraphs}_{\leq 1}$  and the category of FODC  $(\Gamma^1, d)$  on  $\mathbf{k}$ -algebras  $A = \mathbf{k}[V]$ , with  $V$  finite set.

We can extend the definition of FODC from  $\text{diGraphs}_{\leq 1}$  to  $\text{diGraphs}$ , obtaining still an equivalence of categories.

Theorem (F., Simonetti, Zanchetta 2025)

We have a fully faithful contravariant functor

$$\mathcal{F} : \text{diGraphs} \longrightarrow (\text{FODC})_e, \quad G \mapsto (\Gamma_G^1, d)$$

giving an antiequivalence of categories, where  $(\text{FODC})_e$  consists of all the FODC  $(\Gamma^1, d)$  coming from an étale cover of a given graph.

## Definition

Let  $G = (V, E) \in \text{diGraphs}$ ,  $A = \mathbf{k}[V]$  and  $(\Gamma^1, d)$  the FODC on  $A$ . Let  $M$  be a free rank  $n$  left  $A$ -module. We define a **left noncommutative connection**  $\nabla$  on  $M$  as a map

$$\nabla : M \longrightarrow \Gamma^1 \otimes M$$

satisfying the Leibniz identity, i.e:

$$\nabla(fm) = df \otimes m + f\nabla m, \quad f \in A, \quad m \in M$$

Analogously, given a free rank  $n$  right  $A$ -module  $M$ , one can define a **right noncommutative connection**  $\nabla$  on  $M$  as a map

$$\nabla : M \longrightarrow M \otimes \Gamma^1$$

satisfying the Leibniz identity:

$$\nabla(mf) = m \otimes df + (\nabla m)f, \quad f \in A, \quad m \in M$$

Once a basis  $\{e_i\}_{i=1}^n$  for the free  $A$ -module  $M$  is chosen, a non commutative right connection amounts to give a map:

$$e_i f^i \mapsto e_i \otimes df^i + e_j \otimes \omega_i^j f^i$$

where  $\omega_i^j$  is a matrix of 1 forms, i.e. elements of  $\Gamma^1$

## Observation

There is a bijective correspondence between the two notions:

- ① A noncommutative right connection on  $M$ , a right  $A$ -module of rank  $n$ , with respect to the FODC given via  $G$  on  $A$ .
- ② A connection on a digraph.

(2)  $\rightarrow$  (1). In fact, consider a vector bundle  $\mathcal{F}$  of rank  $n$  on  $V$ , a frame bundle

$$V \ni v \mapsto \{e_i^v\}$$

and a free rank  $n$  right  $A$ -module  $n$  with the choice of a basis  $\{e_i\}_{i=1}^n$ .

Then given a connection  $\Theta_{e,u \rightarrow v} := \mathcal{R}_{e,u \rightarrow v} - \mathbb{1}_{v,u}$  we get:

$$\omega_i^j = \sum_{e,x \rightarrow y} [\mathcal{R}_{e,x \rightarrow y,i}^j - \delta_{i,j}] \omega_{e,x \rightarrow y} \quad (9)$$

$$M \ni e^i f_i \mapsto \sum_{e,x \rightarrow y} e_j \otimes [f^i(y) \mathcal{R}_{e,x \rightarrow y,i}^j - f^i(x) \delta_{ij}] \omega_{e,x \rightarrow y} \in M \otimes \Gamma_G^1 \quad (10)$$

(1)  $\rightarrow$  (2). Conversely given a right connection

$$e_i f^i \mapsto e_i \otimes df^i + \sum_{j=1}^n e_j \otimes \omega_i^j f^i$$

where  $\omega_i^j = \sum_{e,x \rightarrow y} a_{e,x \rightarrow y,i}^j \omega_{e,x \rightarrow y}$  (using the basis  $\{\omega_{e,x \rightarrow y}\}$  of  $\Gamma_G^1$  as a  $\mathbf{k}$ -vector space), by setting  $R_{e,x \rightarrow y,i}^j := a_{e,x \rightarrow y,i}^j + \delta_{ij}$  we get a connection.

## Definition

Assume we have:

- $G \in \text{diGraphs}_{\leq 1}$ ,  $(\Gamma_G^1, d)$  a FODC on  $A = \mathbf{k}[G]$
- $S$  a subset of its triangular cliques
- $M$  a free  $A$  bimodule of rank  $n$  with basis  $\{e_i\}_{i=1}^n$ ,  $A = \mathbf{k}[G]$ .
- $\nabla : M \rightarrow \Gamma_G^1 \otimes M$  a noncommutative right connection

We define:

- the **curvature** of  $\nabla$  as  $R_\nabla : M \rightarrow M \otimes \Gamma_G^2$  as the right  $A$  module map defined on the basis  $\{e_i\}_{i=1}^n$  as follows

$$R_\nabla(e_i) = e_j \otimes d\omega_i^j + e_j \otimes \omega_k^j \wedge \omega_i^k$$

- the **curvature outside of  $S$**  of  $\nabla$  as:

$$R_\nabla^S := (\mathbb{1} \otimes \pi_S) \circ R_\nabla : M \rightarrow M \otimes \Gamma_S^2$$

where  $\pi_S : \Gamma_G^2 \rightarrow \Gamma_S^2 = \Gamma_G^2 / \langle S \rangle$  is the projection morphism.

We say that  $\nabla$  is **flat outside of  $S$**  if  $R_\nabla^S = 0$ . We say that  $\nabla$  is **flat** if  $R_\nabla = 0$ .

## Observation

We can rewrite  $R_\nabla$  in terms the weak parallel transport associated with  $\nabla$  as

$$R_\nabla(e_i) = \sum_{x \rightarrow y \rightarrow z \in \text{tri}(G)} (\mathcal{R}_{x \rightarrow y, k}^j \mathcal{R}_{y \rightarrow z, i}^k - \mathcal{R}_{x \rightarrow z, i}^j) e_j \otimes \omega_{x \rightarrow y \rightarrow z} \quad (11)$$

$\text{tri}(G)$  is the set of all triangular cliques of  $G$ .

## Proposition

Let be  $G$ ,  $M$  and  $\nabla$  as above. Then:

1. If  $\nabla$  is flat then  $\mathcal{R}_{x \rightarrow z} = \mathcal{R}_{x \rightarrow y} \mathcal{R}_{y \rightarrow z}$  for each triangular clique. In particular, we have that  $\mathcal{R}_{x \rightarrow y} = \mathcal{R}_{y \rightarrow x}^{-1}$  for all edges  $x \rightarrow y \in E_G$  that are part of a triangular clique of the form  $x \rightarrow y \rightarrow x$ .
2. Assume  $G$  to be bidirected. Consider the set of triangular cliques  $S$  consisting of all triangular cliques of the form  $x \rightarrow y \rightarrow z$  having  $x, y, z \in V_G$  all distinct. Then  $\nabla$  is flat outside of  $S$  if and only if the weak parallel transport associated to  $\nabla$  is a parallel transport i.e.:

$$\mathcal{R}_{x \rightarrow y} = \mathcal{R}_{y \rightarrow x}^{-1} \text{ for all edges } x \rightarrow y \in E_G$$

## Definition

Let  $(\Gamma_G^1, d_G)$  be a FODC on  $\mathbf{k}[V]$  associated to  $G = (V, E) \in \text{diGraphs}$ . We define

- ① a quantum metric on  $\Gamma^1$ , a bimodule map

$$(\cdot, \cdot) : \Gamma_G^1 \otimes_{A_G} \Gamma_G^1 \longrightarrow A_G$$

- ② A  $\mathbf{k}$ -linear map  $\Delta : A_G \rightarrow A_G$  is a second order Laplacian if

$$\Delta(ab) = (\Delta a)b + a\Delta b + 2(da, db)$$

- ③ Graph laplacians associated to the metric  $(\cdot, \cdot)$  are given by:

$$\Delta_\theta(a) := 2(\theta, da) \quad , \quad \theta\Delta(a) := -2(da, \theta)$$

where  $\theta = \sum \omega_{x \rightarrow y}$ .

## Proposition

If we fix the basis  $\{\delta_x\}_{x \in V_G}$  for  $\mathbf{k}[V_G]$ , we identify  $\mathbf{k}[V_G] \cong \mathbf{k}^{|V_G|}$ ,  $L$  is a linear operator and one can readily check:

$$L = D - A = (1/2)\Delta_\theta, \quad \text{for } \lambda_{v \rightarrow w, w \rightarrow v} = \lambda_{w \rightarrow v, v \rightarrow w} = 1$$

where  $D$  is the degree matrix (diagonal matrix with the degree of vertices on the diagonal) and  $A$  is the adjacency matrix of  $G$ .

## Observation

The equality  $L = D - A = 2\Delta_\theta$  is obtained from the comparison of the expressions of  $L$  and  $\Delta_\theta$ :

$$(La)(x) = \sum_{y, (x,y) \in E_G} (a(x) - a(y)), \quad \Delta_\theta a(x) = 2 \sum_{y, x \rightarrow y \in E_G} \lambda_{x \rightarrow y, y \rightarrow x} (a(x) - a(y))$$

We can extend the definition  ${}_\theta\Delta$  when a right connection is given on a vector bundle.

## Definition

Assume we have:

- $G \in \text{diGraphs}_{\leq 1}$  and a FODC  $\Gamma^1$ ,
- $M$  a free rank  $n$  right  $A_G$ -bimodule,
- $\nabla$  a right connection
- $(,)$  a generalized quantum metric on  $\Gamma_G^1$ .

Let  $\eta$  be the left  $A_G$ -module map  $M \otimes_A \Gamma_G^1 \rightarrow M \otimes_A \Gamma_G^1 \otimes_A \Gamma_G^1$ :

$$\eta(m \otimes \omega_{x \rightarrow y}) = m \otimes \omega_{x \rightarrow y} \otimes \theta$$

Define the **connection Laplacian**

$${}_\theta\Delta^M := -2(\mathbb{1} \otimes (,)) \circ \eta \circ \nabla : M \rightarrow M$$

## Observation

Assume we have:

- $G \in \text{diGraphs}_{\leq 1}$  and a FODC  $(\Gamma^1, d)$ ,
- $M$  a free rank  $n$  right  $A_G$ -bimodule, with basis  $\{e_i\}_{i=1}^n$
- $\nabla$  a right connection
- $(, )$  a generalized quantum metric on  $\Gamma_G^1$ .

Then

①

$$\theta \Delta^M(e_i f^i) = -2 \sum_{x \rightarrow y} \lambda_{x \rightarrow y \rightarrow x} (\mathcal{R}_{x \rightarrow y, i}^j f^i(y) - f^i(x)) e_j \delta_x$$

where in the summation only the edges  $x \rightarrow y$  of the maximal bidirected subgraph of  $G$  appear.

② If  $M = A_G$ , we recover the Laplacian  $\theta \Delta$ .



## Observation

*Vector bundles are locally free sheaves (as in ordinary geometry).*

## Theorem (F.-Simonetti-Zanchetta 2025)

*Assume*

- $G \in \text{diGraphs}_{\leq 1}$  is a bidirected graph with  $(\Gamma^1, d)$ , differential calculus
  - $\mathcal{F}$  a vector bundle i.e. a sheaf of vector spaces of rank  $n$  on  $G$
  - $\nabla$  connection with  $\mathcal{R}$  weak parallel transport,
  - $M$  the free right  $A_G$ -module associated to the vector bundle  $\mathcal{F}$ .
- 1 If  $\mathcal{R}^F$  is a parallel transport, then  ${}_{\theta}\Delta^M = -L_F$ .
  - 2 If  $\mathcal{F}$  is a sheaf of inner product spaces and  $F_{v \leq e}^* = F_{v \leq e}^{-1}$  (i.e.  $F$  in an  $O(n)$ -bundle), then

$$\nabla^* \nabla = L_F$$

where we fix isomorphisms  $M \cong M^*$  and  $\Gamma^1 \cong (\Gamma^1)^*$ .

## Remark

*This theorem can be proven more generally in the context of semisimplicial sets and their homology/cohomology.*