Optimizing over iid distributions and the Beat the Average game

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Based on joint work with Pierre C Bellec

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Musings on large numbers

Theorem (Law of large numbers)

Let $X_1, X_2, \ldots$ be iid nonnegative random variables with finite first moment,

$$\mathbb{E}[X_1] < \infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mathbb{E}[X_1]$$

with probability 1.

Intuition: the sample average

$$\frac{1}{n} \sum_{i=1}^{n} X_i$$

does not fluctuate much with n.

Musings on large numbers

- What can we say **without** assuming finite first moment?
- For example, can we bound

$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{\alpha}{m}\sum_{i=1}^{m} X_i\right]$$

  for $n > m$ and $\alpha > 0$?
- Let's look at two special cases:

$$\mathbb{P}\left[X_1 < \frac{X_2 + X_3}{2}\right], \qquad \mathbb{P}[X_1 + X_2 + X_3 < 2X_4].$$

Beat the Average

- In the casino, you can play a new game: you roll a die and the game master rolls two. You win if your number is *at least* the average of the game master's numbers.

- One round costs 1 €, and you get 2 € back if you win.

- Tricky point: the three dice are identical, but exotic!

Beat the Average

- With standard dice, you should play: by symmetry,

$$\mathbb{P}\left[X_1 > \frac{X_2 + X_3}{2}\right] = \mathbb{P}\left[X_1 < \frac{X_2 + X_3}{2}\right],$$

  and there's a nonzero chance of $X_1 = \frac{X_2+X_3}{2}$. Therefore

$$\mathbb{P}[\text{win}] > \frac{1}{2}.$$

- But the dice do not look standard!

- To assume the worst case, you need to determine how large your losing probability can get,

$$\sup_{\mu} \mathbb{P}_{\mu^{\otimes 3}}\left[X_1 < \frac{X_2 + X_3}{2}\right] = ?$$

Beat the Average, card version

- The casino also offers a version with cards: you draw one card from a shuffled deck and the game master draws two. You win if your card is at least the average of the game master's.
- Suppose that the deck has n cards. Then:

Theorem

(a) For any n, the casino can trick you at least as much in the card version as in the dice version.

(b) For $n \to \infty$, the maximal losing probabilities coincide.

For the proof of (a), use a *random* deck!

Beat the Average, card version

Taking $n = 3$ shows that

$$\mathbb{P}\left[X_1 < \frac{X_2 + X_3}{2}\right] \leq \frac{2}{3}.$$

Moreover, $\frac{2}{3}$ is achieved for any n by the deck with values

$$1,\ 2,\ 4,\ \ldots,\ 2^{n-1}.$$

Corollary

$$\boxed{\sup_{\mu} \mathbb{P}_{\mu^{\otimes 3}}\left[X_1 < \frac{X_2 + X_3}{2}\right] = \frac{2}{3}}$$

$\Rightarrow$ You should play neither the card nor the dice game!

Optimizing over iid distributions, harder case

Theorem

$$\boxed{0.400695 \;\leq\; \sup_{\mu} \mathbb{P}_{\mu^{\otimes 4}}[X_1 + X_2 + X_3 < 2X_4] \;\leq\; 0.417}$$

- We conjecture that the explicit construction behind our lower bound is optimal.
- Let's focus on the upper bound.

Harder case

- The "card version" of this problem still gives a convergent hierarchy of upper bounds.
- A "deck of cards" is determined by values

$$X_1,\ \ldots,\ X_n \in \mathbb{R}_+.$$

- An optimal deck is such that the inequality

$$X_i + X_j + X_k < 2X_\ell$$

  is satisfied for as many tuples $ijk\ell \in \{1, \ldots, n\}$ as possible.
- This is a problem of finding a **maximal feasible subsystem** of inequalities.

Determining a maximal feasible subsystem

To approach it, solve the following mixed integer linear program:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{ijk\ell} y_{ijk\ell} \\
\text{with respect to} \quad & y_{ijk\ell} \in \{0,1\} \\
& x_i \in [0,1] \\
\text{subject to} \quad & x_i + x_j + x_k + M(y_{ijk\ell} - 1) \leq 2x_\ell - \varepsilon \\
& 0 \leq x_1 \leq \ldots \leq x_n
\end{aligned}
$$

Here, $\varepsilon \ll 1$ and $M := 3 + \varepsilon$ are constants.

Idea: $y_{ijk\ell} = 1$ indicates that the inequality $x_i + x_j + x_k < 2x_\ell$ is satisfied.

Determining a maximal feasible subsystem

- Problem: a numerical solution does not give a rigorous certificate of optimality.
- Alternative approach: note that

$$X_1 + X_2 + X_5 < 2X_3 \text{ is incompatible with}$$
$$X_3 + X_4 + X_7 < 2X_5$$

and

$$X_1 + X_2 + X_5 < 2X_4 \text{ is incompatible with}$$
$$X_1 + X_4 + X_6 < 2X_5$$
$$X_3 + X_6 + X_8 < 2X_7$$
$$X_4 + X_7 + X_9 < 2X_8.$$

etc.

Determining a maximal feasible subsystem

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$$X_4 + X_7 + X_9 < 2X_8.$$

etc.

- Principle: If we have a collection of N *disjoint* incompatible systems, then at least N inequalities must be violated overall.

A second mixed integer linear program

- We can search for such disjoint collections of systems using a *second* mixed integer linear program!

- To formulate it, let

$$\mathcal{T}^s := \{ijk\ell \mid y_{ijk\ell} = 1\},$$
$$\mathcal{T}^v := \{ijk\ell \mid y_{ijk\ell} = 0\},$$

  where the $y_{ijk\ell}$ are from the first program.

- This will guide us in finding such disjoint collections of systems.

A second mixed integer linear program

This idea can be formalized as follows:

$$
\begin{aligned}
&\text{find any feasible} && y_{tu} \in \{0,1\} && \text{with } (t,u) \in \mathcal{T}^s \times \mathcal{T}^v,\\
&\text{subject to} && \sum_{u \in \mathcal{T}^v} y_{tu} \leq 1 && \text{with } t \in \mathcal{T}^s,\\
& && F(q,u) + \sum_{t \in \mathcal{T}^s} F(q,t) y_{tu} \geq 0 && \text{with } q \in [m],\ u \in \mathcal{T}^v.
\end{aligned}
$$

Here, we write

$$F(q,t) := \delta_{q \leq i} + \delta_{q \leq j} + \delta_{q \leq k} - 2\delta_{q \leq \ell}.$$

A second mixed integer linear program

For $n = 20$, this terminates and we get:

- There are 19380 candidate inequalities.
- Half of them already contradict $0 \leq x_1 \leq \ldots \leq x_{20}$.
- Among the remaining 9690, we find 1614 disjoint incompatible subsystems.
- For each, incompatibility follows by simply adding up the inequalities.
- This gives the rigorous upper bound of

$$\frac{9690 - 1614}{19380} = 0.4167\ldots$$

Lower bound

Consider more generally the maximal probability of an inequality $\sum_i c_i X_i > 0$ under iid distributions.

Proposition

For every ν and $\varepsilon > 0$, there is μ such that

$$\mathbb{P}_{\mu^{\otimes n}}\left[\sum_i c_i X_i > 0\right] \geq \frac{\mathbb{P}_{\nu^{\otimes n}}[\sum_i c_i X_i > 0]}{1 - \mathbb{P}_{\nu^{\otimes n}}[\sum_i c_i X_i = 0]} - \varepsilon.$$

For the proof, take

$$Y_i := X_{i,1} + \eta X_{i,2} + \eta^2 X_{i,3} + \dots$$

for $\eta \ll 1$ and the $X_{i,j}$ iid with law ν.

Lower bound

Applying the proposition with

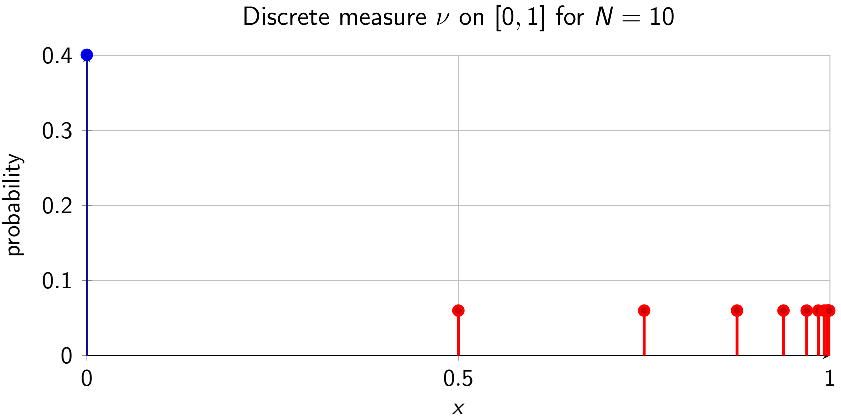
$$\nu := p\delta_0 + \frac{1-p}{N}\sum_{i=1}^{N}\delta_{1-2^{-i}}$$

for $N \gg 1$ results in

$$\sup_{\mu} \mathbb{P}_{\mu^{\otimes n}}[X_1 + X_2 + X_3 < 2X_4] \geq \sup_{p\in(0,1)} \frac{p(2-p)}{1+p+p^2+p^3}.$$

The maximum ≈ 0.400695 is achieved at $p \approx 0.474346$.

Lower bound



Discrete measure ν on $[0, 1]$ for $N = 10$
probability
0.4
0.3
0.2
0.1
0
0
0.5
1
x

AlphaEvolve

MATHEMATICAL EXPLORATION AND DISCOVERY AT SCALE

BOGDAN GEORGIEV, JAVIER GÓMEZ-SERRANO, TERENCE TAO, AND ADAM ZSOLT WAGNER

- AlphaEvolve is a coding agent that automatically evolves code for optimization problems.
- Tested on 67 representative mathematical optimization problems.
- In many cases (e.g. kissing numbers), existing bounds were improved.
- For our $\mathbb{P}[X_1 + X_2 + X_3 < 2X_4]$, it found a lower bound of 0.389.


New result distribution
Visualization of results across 67 problems.
New result
Former new result, got improved upon
Worse than literature bound
Matched known optimal bound
Matched literature bound / N/A
1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67

Summary

- Motivated by the law of large numbers, we have studied how large probabilities like
$$\mathbb{P}\left[\sum_i c_i X_i > 0\right]$$
can get under an iid distribution.

- We have methods to compute upper and lower bounds, but computing exact values is still open.

- Conjecture: an exact computation method based on the fractal nature of near-optimal distributions exists.

- Our $\mathbb{P}[X_1 + X_2 + X_3 < 2X_4]$ has become a benchmark problem for AI mathematics.