Conformally homogeneous Lorentzian spaces

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Let $(M, c = [g])$ be a conformal pseudo-Riemannian manifold.

A diffeomorphism

$$F : M \to M$$

is called a conformal transformation if

$$\forall g \in c \quad F^* g \in c,$$

i.e.,

$$F^* g = e^{2f} g.$$

F is called **non-essential** if

$$\exists h \in c \quad F^* h = h.$$

Otherwise F is **essential**.

$G \subset \mathrm{Conf}(M, c)$ is called **non-essential** if

$$\exists h \in c \quad G \subset \mathrm{Isom}(M, h)$$

Otherwise G is **essential**.

Any Riemannian manifold which admits an essential group of conformal transformations is conformally equivalent to the standard sphere or the Euclidean space (the Lichnerowicz conjecture): Alekseevsky (1972), Obata (1971), Ferrand (1996).

There are many examples of Lorentzian manifolds with essential conformal group: Frances, Melnik, Zeghib,...

Examples of essential conformally homogeneous Lorentzian manifolds: Podoksenov (1992).

Description of Lorentzian manifolds with essential group of homotheties: Alekseevsky (1985)

We study simply connected essential conformally homogeneous conformal Lorentzian manifolds $(M = G/H, c)$.

Two types of such manifolds:

A. Manifolds with non-faithful isotropy representation

$$j : \mathfrak{h} \to \mathfrak{co}(V), \quad V = \mathfrak{g}/\mathfrak{h} = T_o M$$

of the stability subalgebra $\mathfrak{h}$.

B. Manifolds with faithful isotropy representation j.

Alekseevsky (2017): classification of spaces of type A.

Manifolds of type A are conformally flat.

A Lorentzian manifold (M, g) is called a **plane wave** if there exists a vector field p with

$$g(p,p)=0, \quad \nabla p = 0,$$

$$R(X,Y)=0, \quad \nabla_X R = 0 \quad \forall\ X, Y \text{ orthogonal to } p. \tag{1}$$

The metric g of a plane wave may be written locally in the form

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + a_{ij}(u)x^i x^j (du)^2 \tag{2}$$

where $a_{ij}(u)$ is a symmetric matrix of functions. The metric (2) is conformally flat if and only if

$$a_{ij}(u) = \delta_{ij} b(u),$$

where $b(u)$ is a function.

Classification of locally homogeneous plane waves: Blau, O'Loughlin (2003)

Classification of simply connected homogeneous plane waves: Hanounah, Mehidi, Zeghib (**2023**):

(a) the space $\mathbb{R}^{n+2} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ with the metric

$$g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + \left(e^{uF} B e^{-uF}\right)_{ij} x^i x^j (du)^2,$$

(b) the space $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_{>0}$ with the metric

$$g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + \left(e^{\ln(u)F} B e^{-\ln(u)F}\right)_{ij} x^i x^j \frac{(du)^2}{u^2}.$$

Here B and F are respectively symmetric and skew-symmetric matrices. The metrics of type (a) are geodesically complete, while the metrics of type (b) are not geodesically complete.

Each homogeneous plane wave of type (b) is globally conformally diffeomorphic to a homogeneous plane wave of type (a): Holland, Sparling (**2024**)

Indeed, the coordinates transformation

$$v \mapsto v - \frac{1}{4}\sum_{i=1}^{n}(x^i)^2, \quad x^i \mapsto e^{\frac{u}{2}} x^i, \quad u \mapsto e^u,$$

transforms the metric (b) into the metric of the form (a) given by :

$$g = e^u \left(2dvdu + \sum_{i=1}^{n}(dx^i)^2 + \left(e^{uF} \left(B - \frac{1}{4}\,\mathrm{id} \right) e^{-uF} \right)_{ij} x^i x^j (du)^2 \right).$$

Plane wave metric:

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + a_{ij}(u)x^i x^j (du)^2.$$

A homothety transformation of g:

$$(v, x^i, u) \mapsto (\lambda^2 v, \lambda x^i, u), \tag{3}$$

$$g \mapsto \lambda^2 g$$

Theorem 1

Let (M, c) be a simply connected non-conformally flat conformal Lorentzian manifold. Suppose that (M, c) admits an essential transitive group of conformal transformations. Then there exists a metric $g \in c$ such that (M, g) is a complete homogeneous plane wave.

Theorem 2

Let (M, g) *be a simply connected non-conformally flat homogeneous plane wave. Then the group of conformal transformations of* (M, g) *consists of homotheties and is a 1-dimensional extension of the group of isometries.*

Notation

Minkowski space: $V = \mathbb{R}^{1,n+1}$
Witt basis: $p, e_1, \dots, e_n, q$, $(p,q) = 1$, $(p,p) = (q,q) = 0$
$E = \mathbb{R}^n = \mathrm{span}\{e_1, \dots, e_n\}$

$$\wedge^2 V \cong \mathfrak{so}(V) = \mathfrak{so}(1, n+1) :$$

$$(X \wedge Y)Z = (X,Z)Y - (Y,Z)X, \quad \forall X, Y, Z \in V$$

$$\mathfrak{so}(V) = (\mathbb{R}p \wedge q + \mathfrak{so}(E)) + p \wedge E + q \wedge E$$

Isometry Lie algebra of a homogeneous plane wave

$$\mathfrak{isom}(M,g) = \mathfrak{isom}(M,g)_o + V,$$
$$\mathfrak{isom}(M,g)_o = \mathfrak{k} + p \wedge E \subset \mathfrak{so}(V),$$

$$[q,p] = \lambda p, \quad [p,X] = 0, \quad [X,Y] = 0,$$
$$[q, p \wedge X] = p \wedge (\lambda \,\mathrm{id}_E + F)X - X,$$
$$[q, X] = p \wedge BX + FX,$$

for all $X, Y \in E$.

Here $\lambda = 0$ for the spaces of type (a), and $\lambda = 1$ for the spaces of type (b).

$\mathfrak{k} \subset \mathfrak{so}(E)$ the subalgebra commuting with B and F.

Proof of the Main Theorem

Lemma 1

Let $(M = G/H, c)$ be a connected homogeneous conformal manifold. Suppose that a Lie subgroup $\tilde{G} \subset G$ has the open orbit $U = \tilde{G}o = \tilde{G}/\tilde{H}$. If the isotropy group $j(\tilde{H})$ is a subgroup of the orthogonal Lie group $\mathrm{O}(T_oU)$, then the group $\tilde{G}$ preserves the metric $g|_U$ which is the restriction to U of some metric $g \in c$ from the conformal class c.

Lemma 2

Let $M = G/H$ be a connected homogeneous manifold. If a normal subgroup $F \subset G$ has an open orbit $U = Fo$, then F acts on M transitively.

Lemma 3

Let $(M = G/H, c)$ be a homogeneous conformal Lorentzian manifold. Suppose that $F \subset G$ is a normal Lie subgroup of G acting transitively on M by isometries of a metric $g \in c$. Then G acts by homothetic transformations of g.

Let $(M = G/H, c)$ be a simply connected essential conformally homogeneous manifold with faithful isotropy representation

$$j : H \to \mathrm{CO}(V), \quad V = T_o M = \mathfrak{g}/\mathfrak{h}$$

$$\mathfrak{h} \cong j(\mathfrak{h}) \subset \mathfrak{co}(V) = \mathbb{R}\,\mathrm{id}_V \oplus \mathfrak{so}(V)$$

$$\tilde{\mathfrak{h}} := \mathfrak{h} \cap \mathfrak{so}(V)$$

$$D = \mathrm{id}_V + C \in \mathfrak{h}, \quad C \in \mathfrak{so}(V)$$

such that

$$\mathfrak{h} = \mathbb{R}D + \tilde{\mathfrak{h}}$$

identify V with a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} + V = \mathbb{R}D + \tilde{\mathfrak{h}} + V.$$

In general, this decomposition is not reductive, but we may assume that it is invariant with respect to a maximal reductive subalgebra of $\mathfrak{g}$.

Lemma 4

If $\mathrm{id}_V \in \mathfrak{h}$, *then* (M, c) *is conformally flat.*

Proof. Suppose that $\mathrm{id}_V \in \mathfrak{h}$. Then,

$$\mathfrak{h} = \mathbb{R}\,\mathrm{id}_V \oplus \tilde{\mathfrak{h}}.$$

Since $[\mathrm{id}_V, \mathfrak{h}] = 0$, there exists an $\mathrm{ad}_{\mathrm{id}_V}$-invariant subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. This subspace may be identified with V. Since $\mathrm{ad}_{\mathrm{id}_V}$ annihilates $\mathfrak{h}$ and acts on V as an identity, it holds

$$[\tilde{\mathfrak{h}}, V] \subset V, \quad [V, V] = 0.$$

The Lie subgroup $\tilde{G} \subset G$ is normal. Since $\tilde{\mathfrak{g}}$ contains V, the $\tilde{G}$-orbit of the point o is open. By Lemma 2, $\tilde{G}$ acts transitively on M. By Lemma 1, there exists a metric $g \in c$ such that $\tilde{G}$ acts by isometries of g. Since $[V, V] = 0$, the metric g is flat. □

Thus,

$$D = \mathrm{id}_V + C \in \mathfrak{h}, \quad C \in \mathfrak{so}(V), \quad C \neq 0$$

Canonical forms of the elements $C \in \mathfrak{so}(V)$:

Elliptic. C annihilates a time-like vector $e_- \in V$,

$$C = C_0 \in \mathfrak{so}(E^{n+1}) \subset \mathfrak{so}(V), \quad E^{n+1} = e_-^\perp.$$

Hyperbolic. $\exists$ Witt basis $p, e_1, \dots, e_n, q$ of V,

$$C = \alpha p \wedge q + C_0, \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0,$$

$$C_0 \in \mathfrak{so}(E), \quad E = \mathrm{span}\{e_1, \dots, e_n\}.$$

Parabolic. $\exists$ Witt basis $p, e_1, \dots, e_n, q$ of V,

$$C = \alpha p \wedge e_1 + C_0, \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0,$$

$$C_0 \in \mathfrak{so}(E^{n-1}), \quad E^{n-1} = \mathrm{span}\{e_2, \dots e_n\}.$$

Aim: prove that

$$C = -p \wedge q + C_0$$

$D \in \mathfrak{h} \subset \mathfrak{g}$

$$\mathrm{ad}_D[X, Y] = [\mathrm{ad}_D X, Y] + [X, \mathrm{ad}_D Y], \quad \forall X, Y \in \mathfrak{g}$$

we use eigenvalue analysis of ad_D

Demonstration in some cases

Suppose now that $D = \mathrm{id}_V + C \in \mathfrak{h}$, where C is hyperbolic. In that case the Lie algebra

$$\mathbb{R}D + (\tilde{\mathfrak{h}} \cap \mathfrak{so}(E))$$

is compact. Since

$$[\mathbb{R}D + (\tilde{\mathfrak{h}} \cap \mathfrak{so}(E)), \mathfrak{h}] \subset \mathfrak{h},$$

there exists an $\mathbb{R}D + (\tilde{\mathfrak{h}} \cap \mathfrak{so}(E))$-invariant complement V to $\mathfrak{h}$ in $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + V.$$

Suppose that $D = \mathrm{id}_V + C \in \mathfrak{h}$, where $C = \alpha p \wedge q + C_0$ is hyperbolic with $\alpha \neq \pm 1$. It holds

$$[D, p] = (1 - \alpha)p, \quad [D, q] = (1 + \alpha)q, \quad [D, E] \subset E,$$

and the eigenvalues of D acting on E belong to the set $1 + \mathbb{R}i$. The eigenvalues of D acting on $\mathfrak{co}(V)$ belong to the set $(\pm\alpha + \mathbb{R}i) \cup \mathbb{R}i$.

This implies that if $\alpha \notin \left\{\pm\frac{1}{2}, \pm 1, \pm 2\right\}$, then $[V, V] = 0$, i.e, as above, (M, c) is conformally flat.

Case $\alpha = -2$

Analyzing the eigenvalues of D, we see that

$$[\mathfrak{h}, V] \subset V, \quad [V, V] \subset \mathfrak{h}.$$

This means that

$$\mathfrak{g} = \mathfrak{h} + V$$

is a symmetric decomposition. This implies that (M, c) admits a locally symmetric Weyl connection with the holonomy algebra $[V, V] \subset \mathfrak{co}(V)$.

Dikarev, Galaev, Schneider. Recurrent Lorentzian Weyl spaces. J. Geom. Anal. 2024:

any locally symmetric Weyl connection is closed, i.e., its holonomy algebra is contained in $\mathfrak{so}(V)$. This means that

$$[V, V] \subset \tilde{\mathfrak{h}}...$$

Thus,

$$D = \mathrm{id}_V + C, \quad C = -p \wedge q + C_0 \in \mathfrak{h}, \quad C_0 \in \mathfrak{so}(E)$$

Recall that

$$\tilde{\mathfrak{h}} \subset \mathfrak{so}(V), \quad \mathfrak{so}(V) = (\mathbb{R}p \wedge q + \mathfrak{so}(E)) + p \wedge E + q \wedge E.$$

The eigenvalues of ad_D acting on $\mathbb{R}p \wedge q + \mathfrak{so}(E)$, $p \wedge E$, $q \wedge E$ belong respectively to the sets $\mathbb{R}i$, $1 + \mathbb{R}i$, $-1 + \mathbb{R}i$. Since ad_D preserves $\tilde{\mathfrak{h}}$, this implies that

$$\tilde{\mathfrak{h}} = \Big(\tilde{\mathfrak{h}} \cap \big(\mathbb{R}p \wedge q + \mathfrak{so}(E)\big)\Big) + (\tilde{\mathfrak{h}} \cap p \wedge E) + (\tilde{\mathfrak{h}} \cap q \wedge E). \quad (4)$$

If the projection of $\tilde{\mathfrak{h}} \cap \big(\mathbb{R}p \wedge q + \mathfrak{so}(E)\big)$ to $\mathbb{R}p \wedge q$ is non-trivial, then we may change the element D to an element $\mathrm{id}_V + C_1$, where $C_1 \in \mathfrak{so}(V)$ is elliptic; this would imply that (M, c) is conformally flat. Thus,

$$\tilde{\mathfrak{h}} = \big(\tilde{\mathfrak{h}} \cap \mathfrak{so}(E)\big) + (\tilde{\mathfrak{h}} \cap p \wedge E) + (\tilde{\mathfrak{h}} \cap q \wedge E). \quad (5)$$

Theorem 3

Let (M, g) be a homogeneous plane wave. Then

$$\mathfrak{conf}(M, g) = \mathbb{R} + \mathfrak{isom}(M, g)$$

and it consists of homothetic vector fields.

More precisely,

$$\mathfrak{conf}(M, g) = \mathbb{R}D + \mathfrak{isom}(M, g), \quad D = \mathrm{id}_V - p \wedge q.$$

Let again (M, c) be a simply connected conformally homogeneous conformal Lorentzian manifold. We have proved:

$$\mathfrak{g} = \mathbb{R}D + \tilde{\mathfrak{h}} + V, \quad V = \mathbb{R}p + E + \mathbb{R}q,$$

where

$$D = \mathrm{id}_V - p \wedge q + C_0, \quad C_0 \in \mathfrak{so}(E),$$

$$\tilde{\mathfrak{h}} = (\tilde{\mathfrak{h}} \cap \mathfrak{so}(E)) + (\tilde{\mathfrak{h}} \cap p \wedge E) + (\tilde{\mathfrak{h}} \cap q \wedge E),$$

$$[\mathbb{R}D + \tilde{\mathfrak{h}} \cap \mathfrak{so}(E), V] \subset V.$$

Then

$$[V, V] \subset \tilde{\mathfrak{h}} \cap p \wedge E + V.$$

Let

$$p \wedge E_1 = \tilde{\mathfrak{h}} \cap p \wedge E.$$

It holds

$$[p \wedge E_1, V] \subset p \wedge E_1 + V.$$

We conclude that

$$\hat{\mathfrak{f}} = \mathbb{R}D + p \wedge E_1 + V \subset \mathfrak{g}$$

is a subalgebra. The orbit of o for $\hat{F} \subset G$ is an open set U. The subspace

$$\mathfrak{f} = p \wedge E_1 + V \subset \hat{\mathfrak{f}}$$

is an ideal and it contains V. By Lemma 1, there exists a metric g_U on U such that F is a transitive group of isometries of (U, g_U). By Lemma 3, $\hat{F}$ consists of homothetic transformations of g_U.

Lemma 5

The homogeneous Lorentzian manifold $(U = F/F_o, g_U)$ is a homogeneous plane wave.

We obtain the inclusion

$$\mathfrak{g} \hookrightarrow \mathfrak{conf}(U, g_U) = \mathbb{R}D + \mathfrak{k} + p \wedge E + V.$$

Consequently

$$\tilde{\mathfrak{h}} \subset \mathfrak{k} + p \wedge E.$$

We conclude that

$$\mathfrak{h} = \mathbb{R}D + \left(\tilde{\mathfrak{h}} \cap \mathfrak{so}(E)\right) + (\tilde{\mathfrak{h}} \cap p \wedge E).$$

This implies that

$$\mathfrak{f} = p \wedge E_1 + V = (\tilde{\mathfrak{h}} \cap p \wedge E) + V \subset \mathfrak{g}$$

is an ideal, and the subgroup $F \subset G$ is normal. By Lemma 2, $U = M$ and $g = g_U$ is a metric on M from the conformal class c. Thus, (M, g) is a homogeneous plane wave. □