Graded geometry of local gauge theories

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Background

- *Batalin-Vilkovisky* (BV) formalism.
- *Alexandrov, Kontsevich, Schwartz, Zaboronsky (AKSZ)* construction of BV for Lagrangian topological models. Further developments *Cattaneo, Felder, Roytenberg, Reshetikhin, Mnev, Ikeda, . . .*
- BV on jet-bundles, local BRST cohomology *Henneaux, Barnich, Brandt, . . .*
- Unfolded approach in higher spin gauge theories *M.Vasiliev*
- Geometric approach to PDEs *Vinogradov, Tulczyjew, . . .*
- FDA approach to SUGRA *d'Auria, Fre, Castellani, Grassi . . .*
- BRST first-quantized (cf. L_∞) approach to SFT and gauge fields *Zwiebach; Thorn, Bochicchio, Henneaux, Teitelboim, ...*
- *Fedosov* quantization and its variations

AKSZ construction

$(\mathcal{M}, q, \omega)$ - QP-manifold (target space) equipped with:

- $\mathbb{Z}$-degree (ghost number) gh()
- homological v.f. q, $q^2 = 0$, gh$(q) = 1$
- (odd)symplectic structure ω, gh$(\omega) = n - 1$ such that

$\{\mathcal{L}, \mathcal{L}\} = 0$

$$q^2 = 0\,, \qquad L_q\omega = 0$$

It follows: $i_q i_q \omega = 0$ and (locally) $\exists \mathcal{L}$ such that $i_q\omega + d\mathcal{L} = 0$

$(\mathcal{X}, \mathrm{d}_X, \rho)$ (source space)

equipped with $\mathbb{Z}$-degree (ghost number) gh()

homological v.f. d_X and compatible measure ρ

Typically, $\mathcal{X} = T[1]X$, $\dim X = n$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$,

$\mathrm{d}_X = \theta^\mu \frac{\partial}{\partial x^\mu}$, $\mu = 0, \ldots n-1$, and $\rho = 1$

$\theta^\mu \equiv dx^\mu$

$\omega = d\chi$ (on $\mathcal{M}$)

Supermanifold of supermaps: $\hat\sigma : T[1]X \to \mathcal{M}$. ψ^A coordinates on $\mathcal{M}$. Fields: $\psi^A(x,\theta) := \hat\sigma^*(\psi^A)$, $\hat\sigma : T[1]X \to \mathcal{M}$. BV action

$$S_{BV}[\hat\sigma] = \int_{T[1]X} (\hat\sigma^*(\chi)(\mathrm{d}_X) + \hat\sigma^*(\mathcal{L}))\,, \qquad \mathrm{gh}(S_{BV}) = 0$$

χ is the potential: $\omega = d\chi$. In components:

$$S_{BV} = \int d^n x d^n\theta \left[\mathrm{d}_X \psi^A(x,\theta)\,\chi_A(\psi(x,\theta)) + \mathcal{L}(\psi(x,\theta))\right]$$

BV symplectic structure:

$$\bar\omega = \int_{T[1]X} \hat\sigma^*(\omega_{AB})\delta\psi^A(x,\theta)\wedge\delta\psi^B(x,\theta)\,, \qquad \mathrm{gh}(\bar\omega) = -1$$

BV antibracket:

$$\big(F,G\big) = \int_{T[1]X} \frac{\delta^R F}{\delta\psi^A(x,\theta)}\omega^{AB}(\psi(x,\theta))\frac{\delta G}{\delta\psi^B(x,\theta)}\,, \qquad \mathrm{gh}\big(,\big) = 1$$

Master equation:

$$\big(S_{BV}, S_{BV}\big) = 0 \qquad \text{modulo boundary terms}$$

$\delta\varphi^i = R^i_\alpha \varepsilon^\alpha$

$S_{BV} = S_0 + \varphi^*_i R^i_\alpha(\varphi) c^\alpha + \ldots$

Physical fields: those of vanishing ghost degree

$$\psi^A(x,\theta) = \overset{0}{\psi}{}^A(x) + \overset{1}{\psi}{}^A_\mu(x)\theta^\mu + \dots \qquad \mathsf{gh}(\overset{k}{\psi}{}^A_{\mu_1\dots\mu_k}) = \mathsf{gh}(\psi^A) - k$$

If $\mathsf{gh}(\psi^A) = k$ with $k \geqslant 0$ then $\overset{k}{\psi}{}^A_{\mu_1\dots\mu_k}(x)$ is physical. Setting to zero fields of nonzero degree (i.e. restricting to maps) gives the classical action:

$$S[\sigma] = \int_{T[1]X} (\sigma^*(\chi)(\mathsf{d}_X) + \sigma^*(\mathcal{L}))$$

ψ^A - coordinates on $\mathcal{M}$

$q^A(\psi(x,\theta))$.

EL equations of motion:

$$\omega_{AB}(\psi(x,\theta))(\mathsf{d}_X\psi^A - q^A) = 0\,, \quad \Rightarrow \quad (\mathsf{d}_X\psi^A(x,\theta) - q^A(\psi(x,\theta))) = 0$$

provided ω_{AB} is invertible.

More invariantly, if $\psi^A(x,\theta) = \sigma^*(\psi^A)$ the equations of motion read as:

$$\mathsf{d}_X\sigma^*(\psi^A) = \sigma^*(q\psi^A) \qquad \Leftrightarrow \qquad \mathsf{d}_X \circ \sigma^* = \sigma^* \circ q$$

so that σ^* is a morphism of respective complexes. Gauge transformations correspond to trivial morphisms:

$$\delta_\epsilon\sigma^* = \mathsf{d}_X \circ \epsilon^*_\sigma + \epsilon^*_\sigma \circ q$$

ϵ_σ - gauge parameter. $\epsilon^*_\sigma(fg) = (\epsilon^*_\sigma f)\sigma^*(g) + (-1)^{|f|}\sigma^*(f)\epsilon^*_\sigma(g)$, i.e. is a vector field along σ.

Example: CS theory, *AKSZ, 1995*

Target: $\mathcal{M} = \mathfrak{g}[1]$, q-CE differential, ω – invariant form on $\mathfrak{g}$ (degree 2 symplectic structure on $\mathfrak{g}[1]$)
Source: $\mathcal{X} = T[1]X$, dim $X = 3$, gh() – form degree, d_X

$$S_{BV} = \int_X Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) + \text{BV completion}$$

Ghosts and antifields arise as nonzero degree components of a supermap:

$$\widehat{\sigma}^*(C) = \overset{0}{C}(x) + A_\mu(x)\theta^\mu + \frac{1}{2}\overset{2}{C}_{\mu\nu}(x)\theta^\mu\theta^\nu + \frac{1}{6}\overset{3}{C}_{\mu\nu\rho}(x)\theta^\mu\theta^\nu\theta^\rho$$

Introducing $C^*, A^{*\mu}$ via $\overset{3}{C}_{\mu\nu\rho}(x) = \epsilon_{\mu\nu\rho}C^*$ and $\overset{2}{C}_{\mu\nu}(x) = \epsilon_{\mu\nu\rho}A^{*\rho}$ the BV symplectic structure

$$\omega_{BV} = \int_X Tr(\delta A_\mu \wedge \delta A^{*\mu} + \delta C \wedge \delta C^*)$$

$\mathcal{M} = \mathcal{M}_0 \times T^*(\text{ghost space})$

$p, q \quad \Omega \quad c^\alpha$

Example: 1d AKSZ sigma model

Target: BFV phase space $\mathcal{M}$ equipped with symplectic form ω and BFV-BRST charge $\Omega = c^\alpha T_\alpha + \ldots$ such that $\{\Omega, \Omega\} = 0$ and the Hamiltonian $H = H_0 + \ldots$ satisfying $\{H, \Omega\} = 0$. (Generalized) AKSZ action *M.G., Damgaard 2000*

$$S_{BV} = \int dt d\theta (\chi_A \mathsf{d}_X \psi^A - \Omega(\psi(t,\theta)) - \theta H(\psi(x,\theta)))$$

is a BV extension *(Fisch, Henneaux)* of the Hamiltonian action:

$$S_0 = \int dt (p\dot{q} - H_0 - \lambda^\alpha T_\alpha)$$

Lagrange multipliers λ^α arise as 1-forms associated to BFV ghost variables: $\sigma^*(c^\alpha) = \lambda^\alpha(t)\theta$.

The relation between the BV antibracket and BFV Poisson bracket

$$(\cdot,\cdot)_{BV} = \int dt d\theta \{\,\cdot\,,\,\cdot\,\}$$

Explicit realization of the isomorphism of *Barnich, Henneaux 1996*

What we've learned:

– non-diffeo-invariant theories correspond to x^a, θ^a-dependent structures. Suitable language is that of fiber bundles.

– AKSZ unifies BV and BFV. For $X = \Sigma \times \mathbb{R}^1$ taking $T[1]\Sigma$ as a source gives BFV-AKSZ sigma model. *M.G. Barnich 2003; M.G. 2010*. Further developments: *Cattaneo, Mnev, Reshetikhin 2012; Bonechi, Zabzine 2012; . . .* .

– More generally, induces (shifted) BV (BFV) on any source manifold. Gives a natural framework to study gauge theories with (asymptotic) boundaries *M.G, Bekaert 2012; Mnev, Schiavina 2019, MG Markov 2023, . . .*

Towards generalized AKSZ

In general, AKSZ equations of motion

$$\omega_{AB}(\psi(x,\theta))(\mathsf{d}_X\psi^A(x,\theta) - q^A(\psi(x,\theta))) = 0, \quad q^A = q\psi^A\,.$$

For ω_{AB} invertible, these imply (generalized) zero-curvature and hence the system is topological provided M is finite-dimensional.

What about general local gauge theories? Possible way out is infinite-dimensional $\mathcal{M}$ involving all the curvatures. The idea goes back to unfolded approach of *M.Vasiliev*. General formalism and existense: *Barnich, MG, 2010*

An alternative (with $\mathcal{M}$ finite-dim.): take ω degenerate so that AKSZ equations of motion kill only part of the curvature. The first characteristic example is Cartan-Weyl form of Einstein gravity:

ξ – Transl
ρ – Lorentz rotations

Presymplectic AKSZ form of gravity

Target $(\mathfrak{g}[1], q, \omega)$, with $\mathfrak{g}$ Poincare algebra and q its CE differential. Coordinates on $\mathfrak{g}[1]$ in the standard basis ξ^a, ρ^{ab}

$$q\xi^a = \rho^a{}_c\,\xi^c\,, \qquad q\rho^{ab} = \rho^a{}_c\,\rho^{cb}$$

Presymplectic structure: *Alkalaev, M.G. 2013; MG 2016*

$$\omega = \epsilon_{abcd}\xi^a d\xi^b d\rho^{cd}\,, \quad \omega = d\chi$$

$$L_q\omega = 0\,, \qquad d\omega = 0 \qquad \Rightarrow \qquad i_q\omega + d\mathcal{L} = 0$$

AKSZ-like action:

$$S[\sigma] = \int_{T[1]X} \sigma^*(\chi)(\mathrm{d}_X) + \sigma^*(\mathcal{L}) = \int_{T[1]X} (\mathrm{d}_X\gamma^{ab} + \gamma^a{}_c\gamma^{cb})\epsilon_{abcd}e^c e^d$$

where $e^a = \sigma^*(\xi^a)$ and $\gamma^{ab} = \sigma^*(\rho^{ab})$. Familiar Cartan-Weyl action for GR. Generalization for $n > 4$ and $\Lambda \neq 0$ is obvious.

What about the remaining components of supermaps? Full-scale BV formulation?

e – frame $\quad$ γ^{ab} – spin connection.

General axioms:

<u>Def</u> Pre Q-bundle $\pi : (E, Q) \to (\mathcal{X}, q)$ $\mathbb{Z}$-graded manifolds equipped with degree 1 vector fields such that $Q \circ \pi^* = \pi^* \circ q$,
If $Q^2 = 0$ and $q^2 = 0$ one gets $\mathbb{Z}$-graded version of Q-bundle *Kotov, Strobl 2007*.

<u>Def</u> *[MG 22, Dneprov, Gritzaenko, MG 24]* Weak presymplectic gauge PDE is a pre Q-bundle $\pi : (E, Q) \to (T[1]X, \mathrm{d}_X)$ equipped with presymplectic structure ω, $\mathrm{gh}(\omega) = \dim X - 1$ $d\omega = 0$, and a function $\mathcal{L}$, $\mathrm{gh}(\mathcal{L}) = \dim X$:

$$d\omega = 0\,, \qquad \frac{1}{2} i_Q i_Q \omega + Q\mathcal{L} = 0\,, \qquad i_Q\omega + d\mathcal{L} \in \mathcal{I}$$

where $\mathcal{I}$ is the ideal in $\bigwedge^\bullet(E)$ generated by $\pi^*(\alpha)$ with $\alpha \in \bigwedge^{>0}(T[1]X)$.
In coordinates, $\mathcal{I}$ is generated by $dx, d\theta$.

Example: weak presymplectic scalar field

$E = T[1]X \times F$, fiber coordinates:

$$\phi, \phi^a, \qquad \mathrm{gh}(\phi) = \mathrm{gh}(\phi^a) = 0$$

$$Qx^a = \theta^a, \quad Q\theta^a = 0, \quad Q\phi = \theta^a \eta_{ab} \phi^b, \quad Q\phi^a = \theta^a V'(\phi)$$

Presymplectic form (cf. *Kijowski, Tulczyjew 1979, Crnkovic, Witten, 1987,...* presymplectic current):

$$\omega = d\chi, \quad \chi = (\theta^a)_a^{n-1} \phi^a d\phi, \qquad (\theta)_a^{(n-1)} = (*\theta)_a$$

Note: in general $L_Q\omega \neq 0$ and $Q^2 \neq 0$ but the axioms hold!

$$i_Q\omega + d\mathcal{L} \in \mathcal{I} \quad \Longrightarrow \quad \mathcal{L} = -(\theta)^n (\frac{1}{2}\phi_a\phi^a + V(\phi))$$

AKSZ-like (aka intrinsic) action: *Schwinger, De Donder-Weyl*

$$S[\phi, \phi^a] = \int_X (dx)^n \left(\phi^a(\partial_a\phi - \frac{1}{2}\phi_a) - V(\phi)\right)$$

Presymplectic AKSZ form of YM:

$E = T[1]X \times F$, fiber coordinates ($\mathfrak{g}$-valued):

$$C, \quad \text{gh}(C) = 1\,, \qquad F^{a|b}, \qquad \text{gh}(F^{a|b}) = 0$$

$$Qx^a = \theta^a, \qquad Q\theta^a = 0, \qquad QC = -\frac{1}{2}[C, C] + \frac{1}{2}F^{a|b}\theta_a\theta_b, \qquad QF^{a|b} = [F^{a|b}, C]$$

Note $Q^2 \neq 0$, in general. Presymplectic structure satisfying $L_Q\omega \in \mathcal{I}$:

Alkalaev, M.G. 2013

$$\omega = d\chi\,, \qquad \chi = (\theta)^{(n-2)}_{ab} Tr\left(F^{a|b} dC\right)$$

AKSZ-like action ($\sigma^*(C) = A_a(x)\theta^a, \sigma^*(F^{a|b}) = F^{a|b}(x)$):

$$S[\sigma] = \int d^n x\, Tr\left((\partial_a A_b - \partial_b A_a + [A_a, A_b])F^{a|b} - \frac{1}{2}(F^{a|b})^2\right)$$

Features of weak presymplectic gPDEs:

- Almost as good as AKSZ but applies to general local gauge theories

- Encodes a local gauge theory in terms of a finite-dim pre-Q presymplectic manifold that can be regarded as a minimal model of the theory (as we are going to see it arises as a minimal mode of the L_∞ algebra determined by the jet-space BV-BRST differential + descent of the BV symplectic structure)

- Together with minimality condition seems to be an invariant geometrical object underlying local gauge systems. Should be unique modulo suitable equivalence.

- What about full-scale BV? Where does it come from? Existence?

Quasi-regularity

Given a fiber bundle $E \to T[1]X$ one can construct a new bundle $\bar{E} \to X$ whose fiber at $x \in X \subset T[1]X$ is a space of super-maps from $T_x[1]X$ to the fiber over x. Supersections of $\bar{E}$ over X are 1 : 1 with supersections of E over $T[1]X$.
Locally:

$$Smaps(T[1]X, F) \cong Smaps(X, \bar{F}) \,, \qquad \bar{F} = Smaps(T_x[1]X, F))$$

Presymplectic structure ω determines a presymplectic structure $\bar{\omega}_x$ on $\bar{E}_x$ via integration over $T_x[1]X$. We say that ω is quasi-regular if $\bar{\omega}$ is regular. More systematic treatment uses vertical jets.

Thm. [MG22, Dneprov, MG, Gritzaenko 24] Let $(E, Q, T[1]X, \omega)$ be a weak presymplectic gauge PDE. Assume that presymplectic structure ω is quasi-regular. Then, locally,

$$S_{BV}(\widehat{\sigma}) = \int_{T[1]X} (\widehat{\sigma}^*(\chi)(\mathrm{d}_X) + \widehat{\sigma}^*(\mathcal{L}))\,, \quad \omega = d\chi$$

defines a local BV system on the symplectic quotient of $\bar{E}$.

Idea of the proof: Prolongation of Q defines a BV system on the symplectic quotient. Particular case in *Kotov, MG 20*.

Physical explanation: Shifts along $\ker \bar{\omega}$ are algebraic gauge transf. for S_{BV}. Gauge-fixing them gives BV action satisfying BV master-equation modulo boundary terms. In particular, S_{BV} can be used in the path integral

$$\int_{\widetilde{L}} \exp \frac{i}{\hbar} S_{BV}$$

where $\widetilde{L}$ also takes into account $\ker \bar{\omega}$. No need to take the symplectic quotient explicitly

Example: scalar

Recall: fiber coordinates ϕ, ϕ^a Coordinates on $\Gamma_S(E)$:

$$\hat{\sigma}^*(\phi) = \overset{0}{\phi}(x) + \overset{1}{\phi}_a(x)\theta^a + \dots$$
$$\hat{\sigma}^*(\phi^a) = \overset{0}{\phi}{}^a(x) + \overset{1}{\phi}{}^a_b(x)\theta^b + \dots$$

Presymplectic structure $\omega = (\theta^a)^{n-1}_a d\phi^a d\phi$ induced on supermaps:

$$\bar{\omega} = \int_X d^n x \left(\delta\overset{0}{\phi} \wedge \delta\overset{1}{\phi}{}^a_a + \delta\overset{0}{\phi}{}^a \wedge \delta\overset{1}{\phi}_a \right)$$

All the fields are in the kernel except for:

$$\varphi = \overset{0}{\phi}, \quad \varphi^* = \overset{1}{\phi}{}^a_a, \quad \varphi^a = \overset{0}{\phi}{}^a, \quad \varphi^*_a = \overset{1}{\phi}_a$$

Correct set of fields and antifields for the 1st order form of scalar! BV symplectic structure emerged from the presymplectic current!

Example: YM

Recall: fiber coordinates $C, F^{a|b}$. Coordinates on $\Gamma_S(E)$:

$$\hat{\sigma}^*(C) = \overset{0}{C}(x) + A_a(x)\theta^a + \frac{1}{2}\overset{2}{C}_{ab}(x)\theta^a\theta^b \ldots$$

$$\hat{\sigma}^*(F^{a|b}) = \overset{0}{F}{}^{a|b}(x) + \overset{1}{F}{}^{a|b}_c(x)\theta^c + \frac{1}{2}\overset{2}{F}{}^{a|b}_{cd}(x)\theta^c\theta^d + \ldots$$

Presymplectic structure $\omega = \theta^{(2)}_{ab} dF^{ab} dC$ induces on supermaps:

$$\bar{\omega} = \int_X Tr\left(\delta\overset{0}{C} \wedge \delta\overset{2}{F}{}^{a|b}_{ab} + \delta A_a \wedge \delta\overset{1}{F}{}^{a|b}_b + \delta\overset{2}{C}_{ab} \wedge \delta\overset{0}{F}{}^{a|b}\right)$$

All the fields are in the kernel except for:

$$C = \overset{0}{C}, \quad C^* = \overset{2}{F}{}^{a|b}_{ab}, \quad A_a, \quad A^a_* = \overset{1}{F}{}^{a|b}_b, \quad F^{a|b}, \quad F^*_{ab} = \overset{2}{C}_{ab}$$

S_{BV} coincides with the standard BV action for YM in the first-order formalism.

Example: Gravity

Fiber coordinates ξ^a, ρ^{ab}. Coordinates on the supermaps:

$$\hat{\sigma}^*(\xi^a) = \overset{0}{\xi}{}^a(x) + e^a_\mu(x)\theta^\mu + \overset{2}{\xi}{}^a_{\mu\nu}(x)\theta^\mu\theta^\nu + \dots ,$$
$$\hat{\sigma}^*(\rho^{ab}) = \overset{0}{\rho}{}^{ab}(x) + \gamma^{ab}_\mu(x)\theta^\mu + \overset{2}{\rho}{}^{ab}_{\mu\nu}(x)\theta^\mu\theta^\nu + \dots .$$

Prop. *[Kotov, MG 2020]* $\bar{\omega}$ determined by $\omega = \epsilon_{abcd}\xi^a d\xi^b d\rho^{cd}$ is regular provided e^a_μ is invertible.
In particular,

$$S[\hat{\sigma}] = \int \hat{\sigma}^*(\chi)(\mathsf{d}_X) - \hat{\sigma}^*(\mathcal{L})$$

induces a proper BV action on the symplectic quotient.
In contrast to the YM and scalar field examples the symplectic structure is not in Darboux form.

Weak gauge PDEs

MG, Rudinsky 2024

Whats is the analog at the level of equations of motion?

Idea: keep the kernel distribution and forget about the presymplectic structure.

Def. Weak gPDE is a pre-Q-bundle $(E, Q) \to (T[1]X, \mathrm{d}_X)$ equipped with a Q-invariant vertical distribution $\mathcal{K}$ such that $Q^2 \in \mathcal{K}$. gPDE corresponds to $\mathcal{K} = 0$

Thm. Let $(E, Q, T[1]X, \mathcal{K})$ be a weak gPDE. Assume that prolongation $\bar{\mathcal{K}}$ of $\mathcal{K}$ is regular. Then, at least locally, $J^\infty_S(E)/\bar{\mathcal{K}}$ is a local BV system.

The proof is based on the observation: $\bar{Q}^2 \in \bar{\mathcal{K}} \Rightarrow \bar{Q}^2 f = 0$ for any function f such that $\bar{\mathcal{K}} f = 0$.

Any weak presympectic gPDE gives weak gPDE by taking $\mathcal{K} = \left\{V \in \mathrm{Vect}_{\mathrm{v}}(E) : i_V \omega \in \mathcal{I}\right\}$ and forgetting ω.

Example: self-dual YM

$X = \mathbb{R}^4$ with Eucledean metric and $E \to T[1]\mathbb{R}^4$, with the fiber being $\mathfrak{g}[1]$, where $\mathfrak{g}$ is a real Lie algebra. Local coordinates on E are: x^a, θ^a, C^A. Useful convention $C = C^A t_A$. The Q-structure is then defined as

$$Q(x^a) = \theta^a, \quad Q(\theta^a) = 0, \qquad Q(C) = -\frac{1}{2}[C, C]$$

Distribution $\mathcal{K}$ is generated by:

$$K_A^{(1)ab} = \left(\theta^a\theta^b + \frac{1}{2}\epsilon^{ab}{}_{cd}\theta^c\theta^d\right)\frac{\partial}{\partial C^A}, \qquad K_{aA}^{(2)} = \epsilon_{abcd}\theta^b\theta^c\theta^d\frac{\partial}{\partial C^A},$$

Note: $Q^2 = 0$ and $L_Q\mathcal{K} \subset \mathcal{K}$.

Minimal model (in the sense of weak gPDE) of seld-dual YM

Example: self-dual YM

Fields parameterizing the quotient $J^\infty_S(E)/\bar{K}$:

$$\overset{0}{C}, \quad A_a \equiv \overset{1}{C}_{|a}, \quad \mathcal{F}^{*-}_{ab} \equiv \overset{0}{C}_{|ab} - \frac{1}{2}\epsilon_{abcd}\overset{0}{C}^{|cd}.$$

The induced BRST differential s:

$$s(\mathcal{F}^{*-}_{ab}) = -(\mathcal{D}_a A_b - \mathcal{D}_b A_a)^- - [\mathcal{F}^{*-}_{ab}, \bar{C}],$$

$$s(\overset{0}{C}) = -\frac{1}{2}[\overset{0}{C}, \overset{0}{C}], \quad s(A_a) = \mathcal{D}_a \overset{0}{C}$$

where $\mathcal{D}_a = D_a + [A_a, \cdot]$ is the covariant total derivative.

Gives standard BRST complex for self-dual YM.

Where do all these structures come from?

Def Q-manifold (M, Q) (aka dg-manifold) is a $\mathbb{Z}$-graded super-manifold M equipped with the odd nilpotent vector field of degree 1, i.e.

$$Q^2 = 0 , \qquad \text{gh}(Q) = 1$$

$\phi : (M_1, Q_1) \to (M_2, Q_2)$ is a Q-map if $\phi^* \circ Q_2 = Q_1 \circ \phi^*$

Example: $(V[1](\mathcal{M}), Q)$ where $V(\mathcal{M})$ Lie algebroid. Indeed generic Q of degree 1 locally reads as:

$$Q = c^\alpha R_\alpha - \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta}(z) \frac{\partial}{\partial c^\gamma}$$

R_α gives anchor, $U^\gamma_{\alpha\beta}$ bracket, $Q^2 = 0$ encodes compatibility.

Proposition: [AKSZ] Let (M, Q) a Q-manifold, $p \in M$ and $Q|_p = 0$ then $T_p M$ is an L_∞ algebra.

Formal pointed Q-manifolds are 1:1 with L_∞-algebras

Equivalence of Q-manifolds:

Idea: restrict to local analysis. Let

$$M = N \times T[1]V\,, \qquad Q = Q_N + d_{T[1]V}$$

with V a graded space. Then (M, Q) and (N, Q_N) are equivalent. Q-manifold $(T[1]V, d_{T[1]V})$ is called contractible. In coordinates:

$$Q = Q_N + v^\alpha \frac{\partial}{\partial w^\alpha}\,, \qquad Q_N = q^i(\phi)\frac{\partial}{\partial \phi^i}\,.$$

Often one finds a "minimal" equivalent Q-man. In the formal setup this gives a minimal model of the respective L_∞ algebra.

Geometric characterization: let w^a be independent functions such that w^a, Qw^a are also independent then the surface $w^a = 0 = Qw^a$ is a Q-submanifold isomorphic to (N, Q_N). Simple geometric picture of the homotopy transfer

In the context of gauge theories: w^α, v^α – are known as "generalized auxiliary fields" *Henneaux, 1990; Barnich, M.G. 2004*.

Def. [Kotov, Strobl] Locally trivial bundle $\pi : E \to M$ of Q-manifolds is called Q-bundle if π is a Q-map. Section $\sigma : M \to E$ is called Q-section if it's a Q-map.

In general, $\pi : E \to M$ is not a locally trivial Q-bundle.

Indeed, although locally $E \cong M \times F$ (product of manifolds) in general Q is not a product Q-structure of Q_F and Q_M.

Example: let $\pi_X : E \to X$ be a fiber bundle then
$\pi = d\pi_X : (T[1]E, d_E) \to (T[1]X, \mathrm{d}_X)$ is a Q-bundle.

Def. [MG, Kotov] (M, Q) is called an equivalent reduction of (M', Q') if (M', Q') is a locally trivial Q-bundle over (M, Q) with a contractible fiber and (M', Q') admits a global Q-section.

This generates an equivalence relation for Q-manifolds.

Gauge PDEs

Def. Gauge PDE $(E, Q, T[1]X)$ is a Q-bundle over $T[1]X$. In addition: equivalent to nonnegatively graded.

Solutions: $\sigma : T[1]X \to E$ is a solution if

$$\mathsf{d}_X \circ \sigma^* = \sigma^* \circ Q \,, \qquad .$$

Gauge parameter: $Y = Y^A(\psi, x, \theta)\frac{\partial}{\partial \psi^A}$, $\mathsf{gh}(Y) = -1$.

Infinitesimal gauge transformations:

$$\delta_Y \sigma^* = \sigma^* \circ [Q, Y]$$

In a similar way one defines gauge (for gauge)^N symmetries.

In local coordinates $x^\mu, \theta^\mu, \psi^A$:

$$Q = \theta^\mu \frac{\partial}{\partial x^\mu} + Q^A(\psi, x, \theta)\frac{\partial}{\partial \psi^A} \,, \qquad \mathsf{d}_X \psi^A(x, \theta) = Q^A(\psi^A(x, \theta), x, \theta)$$

Equivalence of gauge PDEs

Def. A sub-gPDE $(\tilde{E}, \tilde{Q}, T[1]X) \subset (E, Q, T[1]X)$ (i.e. $\tilde{E} \subset E$ is a subbundle, Q restricts to $\tilde{Q}$) is called an equivalent reduction if E is a locally trivial Q-bundle over $\tilde{E}$ (as bundles over $T[1]X$) with a contractible fiber.

In local coordinates: if in adapted coordinates $x^\mu, \theta^\mu, \phi^i, w^a, v^a$ one has $Qw^a = v^a$ and $\tilde{E}$ is singled out by $w^a = 0 = v^a$ then $\tilde{E}$ is an equivalent reduction.

A version of elimination of "generalized auxiliary fields" *Henneaux, 1990; Barnich, M.G. 2004; M.G. Kotov 2019*.

Example: BV formulation (EOM level)

Let $(J^\infty(\mathcal{E}), s)$ be a local BV system, i.e. $\mathcal{E}$ is the BV bundle (fields, ghosts, antifields) and s, $s^2 = 0$ is the BRST differential on $J^\infty(\mathcal{E})$.

Take $J^\infty(\mathcal{E})$ pulled back to $T[1]X$ as E, total degree as a degree, and $Q = d_{\mathsf{h}} + s$. Locally, the gauge system determined by $(E, Q, T[1]X)$ is equivalent to the one encoded in the BV formulation $(J^\infty(\mathcal{E}), s)$. *Barnich, MG 2010*

The notion of gauge PDE is sufficeintly flexible to include BV as a particular case and hence all reasonable gauge theories. Justifies definition.

Example: PDE

Let $E_0 \to X$ be a bundle equipped with Cartan distribution. Extend to a bundle $E \to T[1]X$, the Cartan distribution defines d_{h} on E:

$$d_{\mathsf{h}} = \theta^a D_a \,, \qquad (\theta^a \equiv dx^a)$$

We arrive at Q-bundle $(E, d_{\mathsf{h}}, T[1]X)$.
Seen as a section of $E \to T[1]X$, a solution is a Q-section. If ψ^A are local fiber coordinates the section is parameterized by $\sigma^A(x) = \sigma^*(\psi^A)$
Q-map condition $\mathrm{d}_X \circ \sigma^* = \sigma^* \circ d_{\mathsf{h}}$ gives:

$$\frac{\partial}{\partial x^a}\sigma^A(x) = \Gamma^A_a(\sigma(x), x)\,, \qquad d_{\mathsf{h}} = \theta^a D_a = \theta^a(\frac{\partial}{\partial x^a} + \Gamma^A_a(\psi, x)\frac{\partial}{\partial \psi^A})$$

cf. “unfolded” representation of *M.Vasiliev*.
Usual PDEs are gauge PDEs whose grading is horizontal

Riemannian geometry as a gauge PDE

Take $G = (T^*X \vee T^*X)_{\mathsf{nd}} \oplus T[1]X$. Take E to be $J^\infty(G)$ pulled back to $T[1]X$. Local trivialization:

$$x^a, \theta^a, \qquad g_{ab}, g_{ab|c}, \ldots, \quad \xi^a, \xi^a{}_{|c} \cdots$$

In a suitable trivialization (cf. AKSZ):

$$Q = \mathsf{d}_X + \gamma\,, \quad \gamma g_{ab} = \xi^c g_{ab|c} + \xi^c{}_{|a} g_{cb} + \xi^c{}_{|b} g_{ac}\,, \quad \gamma \xi^a = \xi^c \xi^a{}_{|c}\,, \ldots$$

E.g. Lagrangians: $H^n(Q, local functions)$, $n = \dim X$. Applies to generic off-shell (equivalent to jets) gauge PDEs.

Locally, $E = (T[1]X, \mathsf{d}_X) \times (\mathcal{F}, q)$, i.e. Locally-trivial Q-bundle.

Minimal model

Restrict to local analysis. $\Gamma^a_{(bc|d\ldots)}$ form contractible pairs with $\xi^a_{bcd\ldots}$ and g_{ab} with symmetric part of ξ^a_b. Resulting minimal model *Stora; Barnich, Brandt, Henneaux; Vasiliev . . .*:

Coordinates: $x^\mu, \theta^\mu, \qquad \xi^a, \rho^a{}_b, \qquad R_{ab}{}^c{}_d, R_{a(b}{}^c{}_{de)}, \ldots, R_{a(b}{}^c{}_{de\ldots)}, \ldots$

$$Qx^\mu = \theta^\mu, \quad Q\xi^a = \rho^a{}_c\,\xi^c, \qquad q\rho^{ab} = \rho^a{}_c\,\rho^{cb} + \lambda\xi^a\xi^b + \xi^c\xi^d R^{ab}_{cd},$$
$$QR_{ab}{}^c{}_d = \xi^e R_{a(b}{}^c{}_{de)} + \rho_a{}^f R_{fb}{}^c{}_d + \ldots, \qquad \ldots$$

For instance $H^0(Q)$ gives Riemannian invariants. On-shell version: R are totally traceless (only Weyl tensors).

Section:

$$\sigma^*(\xi^a) = e^a_\mu(x)\theta^\mu, \quad \sigma^*(\rho^{ab}) = \gamma^{ab}_\mu(x)\theta^\mu, \quad \sigma^*(R_{ab}{}^c{}_d) = \mathsf{R}_{ab}{}^c{}_d(x), \ldots$$

Equations of motion:

$$\mathsf{d}_X e^a + \gamma^a{}_b e^b = 0\,, \quad \mathsf{d}_X \gamma^{ab} + \gamma^a{}_c \gamma^{cb} = e^c e^d \mathsf{R}_{cd}{}^{ab}\,, \quad \ldots$$

Cartan structure equations. Taking a total degree “gh+form degree” is crucial. Frame-like formulations.

On shell version – equivalent form of Einstein equations.

Presymplectic structures on gauge PDE

How to represent a generic local gauge theory as a presymplectic gauge PDE? At nonlagrangian level: total differential $Q = d_{\mathsf{h}} + s$,

$$\text{total degree} = \text{gh} + \text{"horizontal form degree"}$$

At Lagrangian level: in the simplest case where "gauge part of BRST differential" $\gamma^2 = 0$ there is an easy shortcut: descent completion of the Lagrangian $\overset{n}{\mathcal{L}}$

$$\gamma\overset{n}{\mathcal{L}} + d_{\mathsf{h}}\overset{n-1}{\mathcal{L}} = 0\,, \quad \gamma\overset{n-1}{\mathcal{L}} + d_{\mathsf{h}}\overset{n-2}{\mathcal{L}} = 0\,, \quad \ldots$$
$$\ldots \quad \gamma\overset{1}{\mathcal{L}} + d_{\mathsf{h}}\overset{0}{\mathcal{L}} = 0\,, \qquad \gamma\overset{0}{\mathcal{L}} = 0\,.$$

Defines a QP structure on $T^*[n-1](J^\infty(\mathcal{E}))$. Resulting AKSZ model is equivalent to the parameterized version of the initial local gauge theory *MG 2010*

Descent completed BV

More general but technically involved: descent completion of the BV symplectic structure:

Cattaneo Mnev Reshetikhin; Sharapov; MG; Mnev Schiavina...

$$\overset{n}{\omega} = (dx)^n \omega_{AB}(x,\psi^A) d_{\mathsf{V}}\psi^A d_{\mathsf{V}}\psi^B \,,$$

$$L_s\overset{n}{\omega} + d_{\mathsf{h}}\overset{n-1}{\omega} = 0\,, \quad L_s\overset{n-1}{\omega} + d_{\mathsf{h}}\overset{n-2}{\omega} = 0\,, \quad \ldots$$
$$\ldots L_s\overset{1}{\omega} + d_{\mathsf{h}}\overset{0}{\omega} = 0\,, \quad L_s\overset{0}{\omega} = 0$$

with

$$Q = d_{\mathsf{h}} + s, \qquad \omega = \overset{n}{\omega} + \overset{n}{\omega} + \ldots \overset{0}{\omega}$$

one finds $i_Q i_Q \omega = 0$. Results in presymplectic gauge PDE. Taking minimal model and setting to zero variables from the kernel of ω results in the presymplectic minimal model.

Conclusions

- (Finite-dimensional) super-geometrical objects underlying local gauge theories. Minimal models seem unique.
- Generalization and first principle derivation of the AKSZ construction. Can be considered as an extension of AKSZ to generic local theories.
- Determines a "canonical" first-order realization in terms of the fields taking values in the minimal model. Makes manifest the underlying Cartan geometry. Covariant Hamiltonian formalism. Classification?
- Further examples include conformal gravity *(Denprov MG, 2022)*, supergravity *(MG Mamekin, to appear)*.

- Tool to study geometry underlying a given gauge system. Background fields and background independence can be incorporated in the approach *(MG, Dneprov, to appear)*

- In the case of variational systems unifies Lagrangian BV and Hamiltonian BFV formalism, cf. BV/BFV approach of *Cattaneo et all.*

- Gives a geometrically-invariant approach to study boundary values of gauge fields and asymptotic symmetries *Bekaert, M.G. 2012, MG, Markov 2023*. In particular, Fefferman-Graham construction (and tractor calculus) can be seen as a certain gauge PDE. *Bekaert, M.G. Skvortsov 2017*

- Gives a criterion to characterize local theory in terms of its infinite dimensional equation manifold. Possibly ineteresting in the higher-spin theory context.