Interactions of Riemannian geometry, contact topology & spectral theory

Outline.

1) brief Intro to contact topology
2) Riemannian geometry & Contact topology: why & how ?
3) Outlook

1 | Brief intro to (3D) contact topology

1.1: basic objects

Def: 1) a $^{(pos.)}$ contact form on a 3-dim mfd is a 1-form satisfying $\alpha \wedge d\alpha > 0$

contrast with a foliation: $\alpha \wedge d\alpha \equiv 0$

2) a (coorientable) contact structure on a 3-dim mfd is the kernel of a contact form.

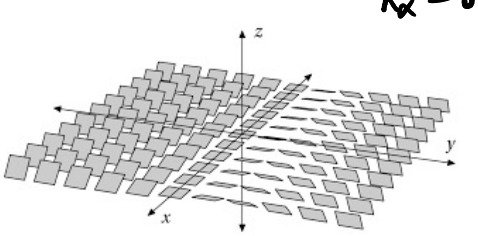
Ex:

·) $M = \mathbb{R}^3$, $\alpha = dz - y\,dx$

$\rightsquigarrow d\alpha = dx \wedge dy$

$\alpha \wedge d\alpha = dx \wedge dy \wedge dz > 0$

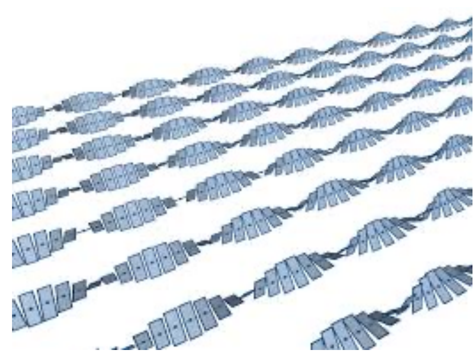
$R_\alpha = \partial_z$



·) $M = T^3$, $\alpha = \sin(z)\,dx - \cos(z)\,dy$

$d\alpha = \cos(z)\,dz \wedge dx + \sin(z)\,dz \wedge dy$

$R_\alpha = \sin(z)\,\partial_x - \cos(z)\,\partial_y$



Lemma: Given α contact form $\exists!$ VF R_α s.t.

$$d\alpha(R_\alpha, -) \equiv 0$$
$$\alpha(R_\alpha) \equiv 1$$

R_α is called the <u>Reeb</u> VF associated to α

Def: (M,α) & (M',α') are regarded as <u>equivalent</u> as contact manifolds ("contactomorphic") if $\exists$ diffeo $\psi: M \longrightarrow M'$ s.t.

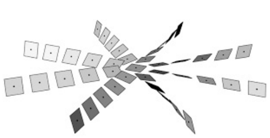
$$\psi^*(\alpha') = f\alpha$$

for some $f \in \mathcal{C}^\infty_{>0}(M)$

$\Leftrightarrow$ $D\psi \cdot (\ker\alpha) = \ker\alpha'$

<u>Darboux Thm</u>: Let (M,α) be a 3d contact mfd. Then $\forall p \in M$ $\exists U_p$ nbhd of p s.t. $(\alpha/_{U_p}, U_p)$ is contactomorphic to a nbhd of 0 in $(dz + \frac{1}{2}r^2 d\theta, \mathbb{R}^3)$

$dz + \frac{1}{2}r^2 d\theta$



i.e. all contact mfds look the same locally

1.2 : Global properties

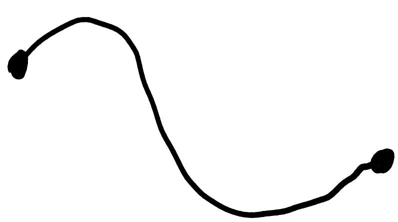
whenever one has a structure would like to have some sort of classification; for this one needs to understand <u>rigidity</u> & <u>flexibility</u> results.

e.g.

<u>Gray stability Thm</u> : Let α_t be a 1-par. family of ctct forms on a cpt mfd. M

Then $\exists$ 1-par. fam. of diffeos ψ_t s.t.

$$\psi_t^* \alpha_t = f_t \alpha_0$$

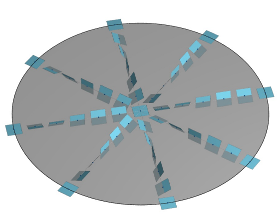


<u>Fundamental dichotomy</u> :

ctct str*

*on closed 3-mfds

overtwisted



$\cos(r)\,dz + r\sin(r)\,d\theta$

tight

2: Riemannian Geometry + Ctct structures: why & how?

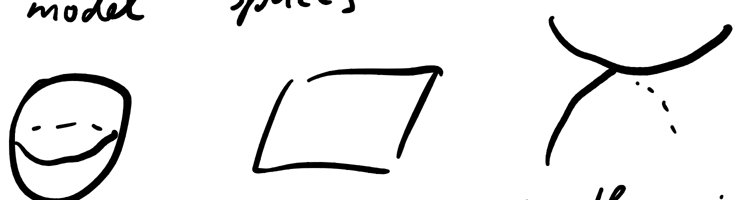
2.1: Why?

⇝ distinction between tight & OT is in terms of twisting ... try to quantify that

⇝ philosophically,

"(Riemannian) geometry dominates in low dimensions"

e.g.: all closed 2-dim surfaces are quotients of model spaces



& some weaker version of this is true in 3D

In particular: geometric analysis solved the 3D Poincaré conjecture

⇝ ∃ interesting geometric flow on 1-forms "Altschuler flow"

2.2 : How ?

Def : a pair (α, g) is called (weakly) compatible if

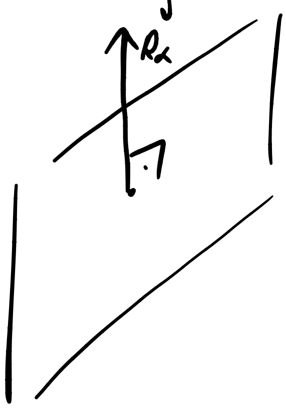
(1) $\|R_\alpha\| = 1$)

2) $R_\alpha \perp_g \ker \alpha$

3) $\alpha \wedge d\alpha = \underline{\lambda}\, dvol_g$, $\lambda \equiv$ const.

"measure of rotation of ξ"

equivalently : $g = g_\xi \oplus_{R_\alpha} \alpha \otimes \alpha$



2 obvious problems : 1) α contact form, not structure !

2) even with α fixed, there is an infinite-dim. space of comp. metrics ... which to choose

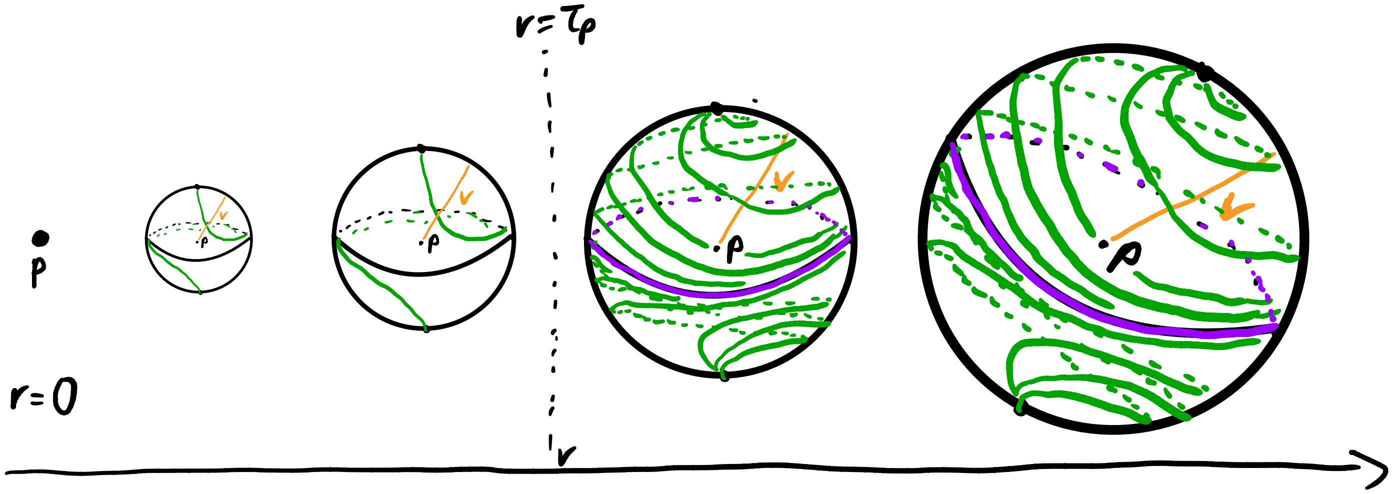
Etnyre, Komendarczyk, Massot

Thm : (α, g) compatible. Then

1) R_α geodesic VF

2) geodesic spheres S_r look like this

(green lines are intersections $\xi_p \cap T_p S_r$)



3) $\tau_p \geq c_p > 0$

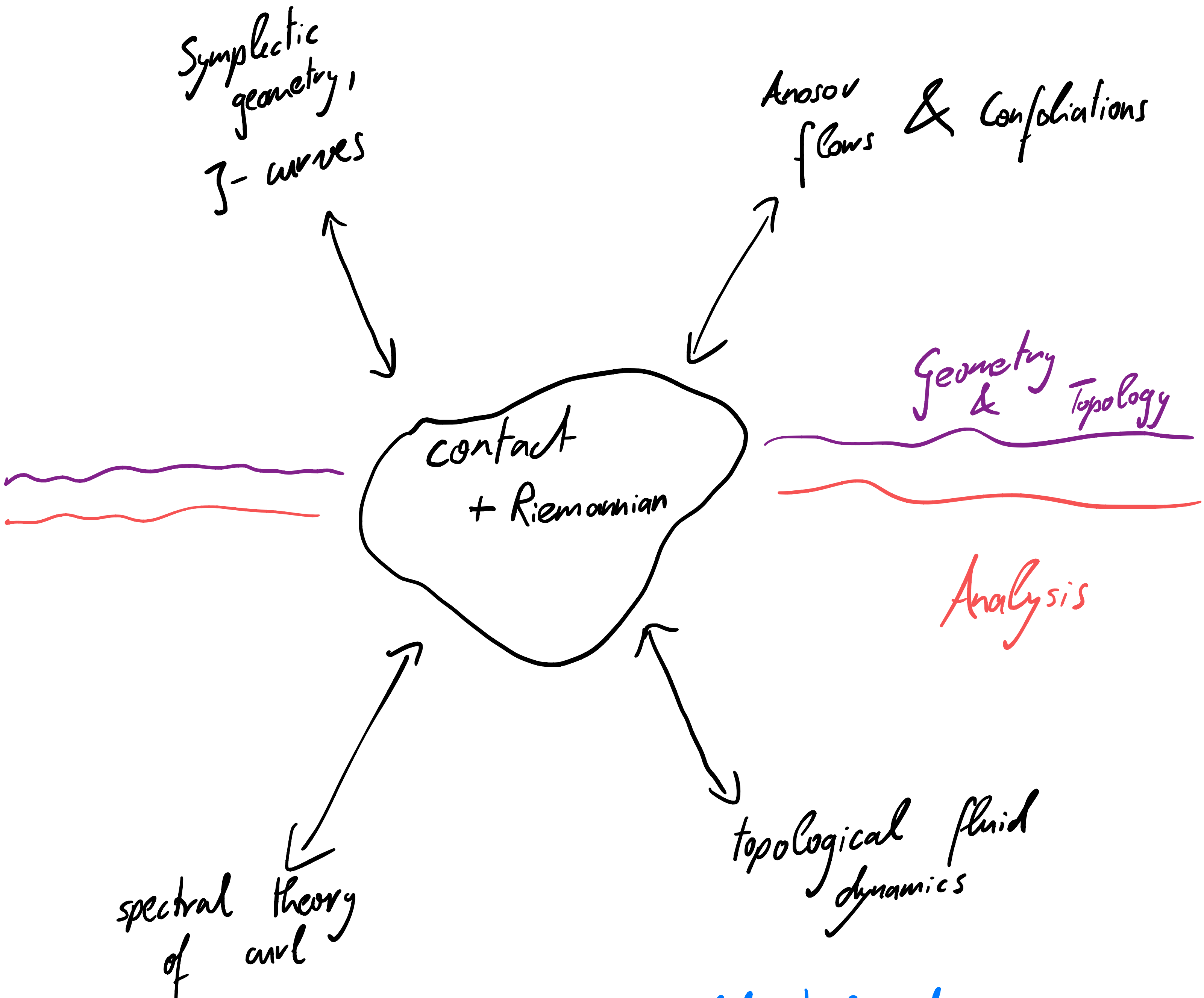
Thm (contact sphere theorem) Etnyre, Komendarczyk, Massot

(M^3, α, g) where (α, g) compatible & g satisfies

$$1 \geq sec(g) > 5/9$$

Then $(\tilde{M}^3, \tilde{\alpha}) \cong_{ctct} (S^3, \alpha_{st})$

3: Outlook (an invitation)



in particular: the most powerful tools of either contact (J-curve based hom. theories) or Riem. (Ricci flow, min. surfaces) have not been employed at all