Representation Theory, Schubert Calculus and Algebraic Topology of Some Embedded Flag Submanifolds

Jakub Knesel

Faculty of Mathematics and Physics, Charles University

18.12.2024

Motivation and Goals

- Understanding K-theory of spaces G/P, where G is complex semisimple Lie group and P a parabolic subgroup of G.
- Complex vector bundles over G/P correspond to P-modules.
- Let $G' \subset G$ be a complex semisimple (sub)group and $P' \subset G'$ a parabolic subgroup s. t. $P' = P \cap G'$. There is a natural embedding $\iota : G'/P' \hookrightarrow G/P$.
- ι induces a short exact sequence of P'-modules/vector bundles:

$$O \longrightarrow \mathrm{T}(G'/P') \longrightarrow \iota^* \mathrm{T}(G/P) \longrightarrow \mathrm{N}(\iota) \longrightarrow O.$$

- $\mathrm{N}(\iota)$ is the normal bundle of the embedding ι and we are interested in its characterictic classes.
- So far we have analyzed the situation $G/P = \mathrm{Gr}(n, k)$.

Table of Contents

CW Structure of $\mathrm{Gr}(n, k)$ - Schubert Cells

- k-dimensional subspace V of $\mathbb{C}^n$ can be represented as a row-space of a $k \times n$ matrix A_V in echelon form. Example given for $\mathrm{Gr}(9, 4)$:

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & 1 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & 0 & * & 1 & 0 \end{pmatrix}$$

- To $V \in \mathrm{Gr}(n, k)$ assign indices $1 \leq t_1 < \ldots < t_k \leq n$ s. t. the t_i-th column of A_V is the i-th canonical vector of $\mathbb{C}^k$.
- $V, W \in \mathrm{Gr}(n, k)$ belong to the same Schubert cell, iff the indices $t_1, \ldots, t_k$ are the same for both A_V and A_W.

Schubert Cells and Partitions

- Schubert cells of $\mathrm{Gr}(n, k)$ correspond to partitions of length k with maximal element $\leq n - k$.
- Schubert cell defined by the sequence $1 \leq t_1 < \ldots < t_k \leq n$ corresponds to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_i = n - k - t_i + i$. Denote this cell by C_λ.
- For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$; $\lambda_1 \leq n - k$; let $\lambda^c := (n - k - \lambda_k, n - k - \lambda_{k-1}, \ldots, n - k - \lambda_1)$.
- Schubert cells C_λ s. t. $|\lambda| = l$ form a free basis of $\mathrm{H}_{2k(n-k)-2l}(\mathrm{Gr}(n, k); \mathbb{Z})$.
- Denote by σ_λ the Poincaré dual of the class C_λ. Then $\{\sigma_\lambda; |\lambda| = l\}$ is a free basis of $\mathrm{H}^{2l}(\mathrm{Gr}(n, k); \mathbb{Z})$. Moreover, $\sigma_\lambda = C^*_{\lambda^c}$ (the Hom-dual of C_{λ^c}).

Grassmannian as a Homogeneous Space

- Left multiplication by matrices from $\mathrm{SL}(n,\mathbb{C})$ on $\mathbb{C}^n$ induces a transitive action of $\mathrm{SL}(n,\mathbb{C})$ on $\mathrm{Gr}(n,k)$.
- Let $\{e_1,\dots,e_n\}$ be the canonical basis of $\mathbb{C}^n$. The stabilizer of $\langle e_{n-k+1},\dots,e_n\rangle$ is the subgroup $P_{n,k}$ consisting of matrices

$$\begin{pmatrix} p_1 & 0 \\ * & p_2 \end{pmatrix}$$

with $p_1 \in \mathrm{GL}(n-k,\mathbb{C}), p_2 \in \mathrm{GL}(k,\mathbb{C}), \det p_1 \det p_2 = 1.$
- Thus, $\mathrm{Gr}(n,k) \simeq \mathrm{SL}(n,\mathbb{C})/P_{n,k}$.
- Representation $\rho : P_{n,k} \longrightarrow \mathrm{GL}(d,\mathbb{C})$ defines a complex vector bundle with total space $\mathrm{SL}(n,\mathbb{C}) \times_\rho \mathbb{C}^d$ defined as

$$\mathrm{SL}(n,\mathbb{C}) \times \mathbb{C}^d/[(gp,v) \sim (g,\rho(p)v); p \in P_{n,k}].$$

The Grassmannian Lattice

$$
\begin{array}{ccccccc}
\mathbb{C}P^1 & \xrightarrow{\eta_{2,1}} & \mathrm{Gr}(3,2) & \xrightarrow{\eta_{3,2}} & \mathrm{Gr}(4,3) & \xrightarrow{\eta_{4,3}} & \dots \\
\alpha_{2,1}\downarrow & & \alpha_{3,2}\downarrow & & \alpha_{4,3}\downarrow & & \\
\mathbb{C}P^2 & \xrightarrow{\eta_{3,1}} & \mathrm{Gr}(4,2) & \xrightarrow{\eta_{4,2}} & \mathrm{Gr}(5,3) & \xrightarrow{\eta_{5,3}} & \dots \\
\alpha_{3,1}\downarrow & & \alpha_{4,2}\downarrow & & \alpha_{5,3}\downarrow & & \\
\mathbb{C}P^3 & \xrightarrow{\eta_{4,1}} & \mathrm{Gr}(5,2) & \xrightarrow{\eta_{5,2}} & \mathrm{Gr}(6,3) & \xrightarrow{\eta_{6,3}} & \dots \\
\alpha_{4,1}\downarrow & & \alpha_{5,2}\downarrow & & \alpha_{6,3}\downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

The Vertical Maps

- $\alpha_{n,k} : \mathrm{Gr}(n,k) \hookrightarrow \mathrm{Gr}(n+1,k)$ is induced by the linear inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ as the last n coordinates.
- It is also induced by

$$\alpha_{n,k} : g \in \mathrm{SL}(n,\mathbb{C}) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \in \mathrm{SL}(n+1,\mathbb{C}).$$

- $\alpha_{n,k}$ maps a Schubert cell C to the Schubert cell $(C \,|\, 0)$ and the corresponding partition $(\lambda_1, \dots, \lambda_k)$ to $(\lambda_1 + 1, \dots, \lambda_k + 1)$.

The Horizontal Maps

- $\eta_{n,k} : \mathrm{Gr}(n,k) \hookrightarrow \mathrm{Gr}(n+1,k+1)$ maps $V \in \mathrm{Gr}(n,k)$ to $\nu(V) \oplus \langle e_{n+1} \rangle$, where $\nu : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ as the first n coordinates and $\{e_1, \dots, e_n, e_{n+1}\}$ is the canonical basis of $\mathbb{C}^{n+1}$.
- It is induced by
$$\eta_{n,k} : g \in \mathrm{SL}(n,\mathbb{C}) \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(n+1,\mathbb{C}).$$
- $\eta_{n,k}$ maps a Schubert cell C to the Schubert cell $\begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$ and the corresponding partition $(\lambda_1, \dots, \lambda_k)$ to $(n-k, \lambda_1, \dots, \lambda_k)$.

Homeomorphism $\mathrm{Gr}(n,k) \simeq \mathrm{Gr}(n,n-k)$

- Let $S = (\delta_{i,n+1-j})_{1\leq i,j\leq n}$. Then S defines a symmetric non-degenerate bilinear form on $\mathbb{C}^n$. Denote by $\perp_S$ orthogonality w. r. t. this bilinear form.
- Define $\Phi_{n,k} : \mathrm{Gr}(n,k) \longrightarrow \mathrm{Gr}(n,n-k)$ by $\Phi_{n,k}(V) = V^{\perp_S}$.
- $\Phi_{n,k}$ is induced by the (outer) automorphism of $\mathrm{SL}(n,\mathbb{C})$ given by $\Phi_{n,k} : g \longmapsto S(g^T)^{-1}S$. We have $\Phi_{n,k}(P_{n,k}) = P_{n,n-k}$ and $\Phi_{n,n-k} \circ \Phi_{n,k} = \mathrm{id}$.
- $\Phi_{n,k}$ is a cellular map and the image of any Schubert cell of $\mathrm{Gr}(n,k)$ is a Schubert cell of $\mathrm{Gr}(n,n-k)$.
- Thus, the maps $(\Phi_{n,k})_*$ and $\Phi^*_{n,k}$ in (co)homology "permute" the free basis given by Schubert cells and their Poincaré duals. In particular, $\Phi^*_{n,k}(\sigma_{(1,\ldots,1,0,\ldots,0)}) = \sigma_{(j,0,\ldots,0)}$.
- At the level of partitions/Young diagrams $\Phi_{n,k}$ acts by transposing the diagrams.

Associated Cohomology Lattice of Grassmannians

Denote $A^j_{n,k} := \mathrm{H}^{2j}(\mathrm{Gr}(n,k);\mathbb{Z})$. We have the j-th cohomology lattice

$$\begin{array}{ccccccc}
A^j_{2,1} & \longleftarrow & A^j_{3,2} & \longleftarrow & A^j_{4,3} & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
A^j_{3,1} & \longleftarrow & A^j_{4,2} & \longleftarrow & A^j_{5,3} & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
A^j_{4,1} & \longleftarrow & A^j_{5,2} & \longleftarrow & A^j_{6,3} & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Cohomology Lattice II

Fix $j \geq 1$. Partitions of j admit a lexicographic ordering, w. r. t. which $(1, 1, \ldots, 1)$ is the first and $(j, 0, \ldots, 0)$ is the last. Denote by R_j the number of all partitions of j. Let $\sigma^j_1, \sigma^j_2, \ldots, \sigma^j_{R_j}$ be the generators of $A^j_{2j,j}$ corresponding to these partitions.

Lemma

(1) The j-th cohomology lattice stabilizes below and to the right from $A^j_{2j,j}$, i. e. $A^j_{n,k} \simeq A^j_{2j,j}$ iff $n \geq 2j$ and $k \geq j$.
(2) The projections in the following diagram are the obvious ones:

$$\begin{array}{ccc}
A^j_{2j-2,j-1} = \bigoplus_{s=2}^{R_j-1} \mathbb{Z}\sigma^j_s & \longleftarrow & A^j_{2j-1,j} = \bigoplus_{s=1}^{R_j-1} \mathbb{Z}\sigma^j_s \\
\uparrow & & \uparrow \\
A^j_{2j-1,j-1} = \bigoplus_{s=2}^{R_j} \mathbb{Z}\sigma^j_s & \longleftarrow & A^j_{2j,j} = \bigoplus_{s=1}^{R_j} \mathbb{Z}\sigma^j_s
\end{array}$$

Cohomology Lattice III

Lemma

(3) Let $\alpha = \alpha_{2j-1,j} \circ \ldots \circ \alpha_{j+1,j} : \mathrm{Gr}(j+1, j) \hookrightarrow \mathrm{Gr}(2j, j)$ *and* $\eta = \eta_{2j-1,j-1} \circ \ldots \circ \eta_{j+1,1} : \mathbb{C}\mathrm{P}^j \hookrightarrow \mathrm{Gr}(2j, j)$. *Then*

$$\begin{aligned} \operatorname{Ker} \alpha^* &= \bigoplus_{s=2}^{R_j} \mathbb{Z}\sigma_s^j \ , \quad \operatorname{Im} \alpha^* = \mathbb{Z}\sigma_1^j, \\ \operatorname{Ker} \eta^* &= \bigoplus_{s=1}^{R_j-1} \mathbb{Z}\sigma_s^j \ , \quad \operatorname{Im} \eta^* = \mathbb{Z}\sigma_{R_j}^j. \end{aligned}$$

I. e., there are short exact sequences

$$O \longrightarrow A^j_{2j-1,j-1} \longrightarrow A^j_{2j,j} \xrightarrow{\alpha^*} A^j_{j+1,j} \longrightarrow O,$$

$$O \longrightarrow A^j_{2j-1,j} \longrightarrow A^j_{2j,j} \xrightarrow{\eta^*} A^j_{j+1,1} \longrightarrow O.$$

Cohomology Lattice IV

For example, if $j = 3$, there are exactly three partitions of 3, namely $(1,1,1),(2,1,0),(3,0,0)$. Only $(2,1,0)$ defines a 6-cell of $\mathrm{Gr}(4,2)$. The H^6-lattice is

$$
\begin{array}{ccccccccc}
O & \longleftarrow & O & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
O & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Cohomology Lattice V

- (1) follows from the fact that Young diagrams of all partitions of j fit inside a $j \times j$ square.
- (2) follows from the fact that the only two partitions of j that do not fit inside the $(j-1) \times (j-1)$ square are the row and column of length j.
- (3) follows from the fact that $\mathbb{CP}^j$ only has partitions with one-row-diagrams and $\mathrm{Gr}(j+1, j)$ only has partitions with one-column-diagrams.

$$\begin{array}{ccccc} & & & & \mathrm{Gr}(j+1,j) \\ & & & & \downarrow \\ & & \mathrm{Gr}(2j-2,j-1) & \xrightarrow{\eta_{2j-2,j-1}} & \mathrm{Gr}(2j-1,j) \\ & & \downarrow \alpha_{2j-2,j-1} & & \downarrow \alpha_{2j-1,j} \\ \mathbb{CP}^j & \longrightarrow & \mathrm{Gr}(2j-1,j-1) & \xrightarrow{\eta_{2j-1,j-1}} & \mathrm{Gr}(2j,j) \end{array}$$

Chern Classes

Theorem

There is a unique sequence of functions $c_1, c_2, \ldots$ which to a complex vector bundle E over a paracompact base space X assign cohomology classes $c_j(E) \in \mathrm{H}^{2j}(X;\mathbb{Z})$ such that
(1) $f^*(c_j(E)) = c_j(f^*E)$ for a continuous map $f : Y \longrightarrow X$.
(2) $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$, where $c(E) = 1 + \sum_{j=1}^{\infty} c_j(E)$.
(3) $c_j(E) = 0$ for all $j > \operatorname{rank} E$.
(4) $c_1(L) = \lambda$, where L is the canonical line-bundle over $\mathbb{CP}^\infty$ and λ is a specified generator of $\mathrm{H}^2(\mathbb{CP}^\infty;\mathbb{Z})$.

We now choose $c_1(L) = -\sigma_1^1$ for all further calculations.

Vector Bundles over $\mathrm{Gr}(n,k)$

- Vector bundles over $\mathrm{Gr}(n,k)$ correspond to representations of $P_{n,k} = \begin{pmatrix} \mathrm{GL}(n-k,\mathbb{C}) & 0 \\ * & \mathrm{GL}(k,\mathbb{C}) \end{pmatrix} \subset \mathrm{SL}(n,\mathbb{C})$.
- $\mathrm{TGr}(n,k)$ is defined by the rep. $\rho_{n,k}$ of $P_{n,k}$ on $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^{n-k}) \simeq \mathfrak{sl}(n,\mathbb{C})/\mathfrak{p}_{n,k}$ given by

$$\rho_{n,k}\left(\begin{pmatrix} p_1 & 0 \\ * & p_2 \end{pmatrix}\right)v = p_1 v p_2^{-1}$$

and is induced by the adjoint rep.

The Normal Bundles

- The maps $\alpha_{n,k}$ and $\eta_{n,k}$ induce

$$O \longrightarrow \mathrm{TGr}(n,k) \longrightarrow \alpha_{n,k}^{*}\mathrm{TGr}(n+1,k) \longrightarrow \mathrm{N}(\alpha_{n,k}) \longrightarrow O,$$

$$O \longrightarrow \mathrm{TGr}(n,k) \longrightarrow \eta_{n,k}^{*}\mathrm{TGr}(n+1,k+1) \longrightarrow \mathrm{N}(\eta_{n,k}) \longrightarrow O.$$

- $\operatorname{rank}\mathrm{N}(\alpha_{n,k}) = k$ and $\mathrm{N}(\alpha_{n,k})$ is defined by the rep.

$$\varphi_{n,k}(\begin{pmatrix} p_1 & 0 \\ * & p_2 \end{pmatrix})v = (p_2^T)^{-1}v \; ; v \in \mathbb{C}^k.$$

- $\operatorname{rank}\mathrm{N}(\eta_{n,k}) = n-k$ and $\mathrm{N}(\eta_{n,k})$ is defined by the rep.

$$\psi_{n,k}(\begin{pmatrix} p_1 & 0 \\ * & p_2 \end{pmatrix})w = p_1 w \; ; w \in \mathbb{C}^{n-k}.$$

The Normal Bundles - Properties

Lemma

(a) $\alpha_{n,k}^{*}\mathrm{N}(\alpha_{n+1,k}) = \mathrm{N}(\alpha_{n,k})$,
(b) $\eta_{n,k}^{*}\mathrm{N}(\eta_{n+1,k+1}) = \mathrm{N}(\eta_{n,k})$,
(c) $\alpha_{n,k}^{*}\mathrm{N}(\eta_{n+1,k}) = \mathrm{N}(\eta_{n,k}) \oplus \mathbb{C}$,
(d) $\eta_{n,k}^{*}\mathrm{N}(\alpha_{n+1,k+1}) = \mathrm{N}(\alpha_{n,k}) \oplus \mathbb{C}$,
(e) $\Phi_{n,k}^{*}\mathrm{N}(\alpha_{n,n-k}) \simeq \mathrm{N}(\eta_{n,k})$,
(f) $\Phi_{n,k}^{*}\mathrm{N}(\eta_{n,n-k}) \simeq \mathrm{N}(\alpha_{n,k})$.
(g) $\mathrm{TGr}(n,k) \simeq \mathrm{N}(\alpha_{n,k}) \otimes \mathrm{N}(\eta_{n,k})$.
(h) There is a short exact sequence

$$O \longrightarrow \mathrm{N}(\alpha_{n,k})^{D} \longrightarrow \mathbb{C}^{n} \longrightarrow \mathrm{N}(\eta_{n,k}) \longrightarrow O.$$

Chern Classes of $\mathrm{N}(\alpha_{n,k})$ and $\mathrm{N}(\eta_{n,k})$

Theorem

(a) $c(\mathrm{N}(\alpha_{n,k})) = 1 + \sigma_1^1 + \sigma_1^2 + \ldots + \sigma_1^k$.
(b) $c(\mathrm{N}(\eta_{n,k})) = 1 + \sigma_1^1 + \sigma_{R_2}^2 + \ldots + \sigma_{R_{n-k}}^{n-k}$.

Proof

- First, $c(\mathrm{N}(\alpha_{n,1})) = c(\mathcal{O}_{\mathbb{CP}^{n-1}}(1)) = 1 + \sigma_1^1$.
- Let $k > 1$ and assume $c(\mathrm{N}(\alpha_{n,k-1})) = 1 + \sum_{j=1}^{k-1} \sigma_1^j$.
- $\eta_{2k-1,k-1}^*(c(\mathrm{N}(\alpha_{2k,k}))) = c(\mathrm{N}(\alpha_{2k-1,k-1}) \oplus \mathbb{C}) = 1 + \sum_{j=1}^{k-1} \sigma_1^j$.
- This determines $c_j(\mathrm{N}(\alpha_{2k,k}))$ for $1 \le j \le k-1$ because of the stability of $2j$-th cohomology lattice ($A_{2k,k}^j = A_{2k-1,k-1}^j = A_{2j,j}^j$).

Chern Classes of $\mathrm{N}(\alpha_{n,k})$ and $\mathrm{N}(\eta_{n,k})$ II

- Moreover, since $c_k(\mathrm{N}(\alpha_{2k-1,k-1})) = 0$, we have $c_k(\mathrm{N}(\alpha_{2k,k})) \in \operatorname{Ker} \eta^*_{2k-1,k-1} = \mathbb{Z}\sigma_1^k$.
- $(\alpha_{2k-1,k} \circ \ldots \circ \alpha_{k+1,1})^* \mathrm{N}(\alpha_{2k,k}) = \mathrm{N}(\alpha_{k+1,k})$ and $(\alpha_{2k-1,k} \circ \ldots \circ \alpha_{k+1,1})^*|_{\mathbb{Z}\sigma_1^k}$ is an isom. onto $A^k_{k+1,k}$. Thus, it is enough to find $c_k(\mathrm{N}(\alpha_{k+1,k}))$.
- $\Phi^*_{k+1,k}\mathrm{N}(\alpha_{k+1,k}) \simeq \mathrm{N}(\eta_{k+1,1})$ and we have the short exact sequence

$$O \longrightarrow \mathcal{O}_{\mathbb{CP}^k}(-1) \longrightarrow \mathbb{C}^{k+1} \longrightarrow \mathrm{N}(\eta_{k+1,1}) \longrightarrow O.$$

Chern Classes of $\mathrm{N}(\alpha_{n,k})$ and $\mathrm{N}(\eta_{n,k})$ III

- $\mathrm{H}^*(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) = \mathbb{Z}[\sigma_1^1]/(\sigma_1^1)^{k+1}$, $c(\mathcal{O}_{\mathbb{C}\mathrm{P}^k}(-1)) = 1 - \sigma_1^1$ and we must have

$$1 = c(\mathbb{C}^{k+1}) = (1-\sigma_1^1)\cdot c(\mathrm{N}(\eta_{k+1,1})).$$

- We compute $c_k(\mathrm{N}(\eta_{k+1,1})) = (\sigma_1^1)^k = \sigma_{R_k}^k$.
- Then $c_k(\mathrm{N}(\alpha_{k+1,k})) = c_k(\Phi_{k+1,k}^*\mathrm{N}(\eta_{k+1,1})) = \Phi_{k+1,k}^*(\sigma_{R_k}^k) = \sigma_1^k$.
- Thus, also $c_k(\mathrm{N}(\alpha_{2k,k})) = \sigma_1^k$ and previous lemma together with stability of cohomology lattices determines $c(\mathrm{N}(\alpha_{n,k})) = c(\mathrm{N}(\alpha_{2k,k}))$ for all n.

Chern Classes of $\mathrm{N}(\alpha_{n,k})$ and $\mathrm{N}(\eta_{n,k})$ IV

- This proves $c(\mathrm{N}(\alpha_{n,k})) = 1 + \sigma_1^1 + \sigma_1^2 + \ldots + \sigma_1^k$.
- Then

$$\begin{aligned} c(\mathrm{N}(\eta_{n,k})) &= c(\Phi_{n,k}^* \mathrm{N}(\alpha_{n,n-k})) \\ &= \Phi_{n,k}^*(1 + \sum_{j=1}^{n-k} \sigma_1^j) \\ &= 1 + \sum_{j=1}^{n-k} \sigma_{R_j}^j. \end{aligned}$$

□

The Normal Bundle of $\iota : \mathrm{Gr}(n,k) \longrightarrow \mathrm{Gr}(n',k')$

- Let $\iota : \mathrm{Gr}(n,k) \longrightarrow \mathrm{Gr}(n',k')$ be the composition $\eta_{n'-1,k'-1} \circ \ldots \circ \eta_{n'-k'+k,k} \circ \alpha_{n'-k'+k-1,k} \circ \ldots \circ \alpha_{n,k}$, so that $n \leq n'$, $k \leq k'$.
- We have a short exact sequence

$$O \longrightarrow \mathrm{TGr}(n,k) \longrightarrow \iota^*\mathrm{TGr}(n',k') \longrightarrow \mathrm{N}(\iota) \longrightarrow O.$$

- Applying the pullback rules successively, we obtain

$$\mathrm{N}(\iota) \simeq \left[\bigoplus^{(n'-k')-(n-k)} \mathrm{N}(\alpha_{n,k})\right] \oplus \left[\bigoplus^{k'-k} \mathrm{N}(\eta_{n,k})\right] \oplus \mathbb{C}^m,$$

where $m = ((n'-k')-(n-k))(k'-k)$.

Pieri Formula

Theorem

Let $0 \leq l \leq n-k$, λ the partition $(l, 0, \dots, 0)$ and μ a partition of length k and maximal element $\leq n-k$. Then $\sigma_\lambda \smile \sigma_\mu = \sum_\nu \sigma_\nu$, where we sum over all partitions ν that can be obtained by adding one box to exactly l different columns of the Young diagram of μ.

- Since $\Phi^*_{n,k}$ acts on partitions by transposing their Young diagrams, we also obtain the "dual" Pieri formula:

Theorem

Let $0 \leq l \leq n-k$, λ the partition $(1, \dots, 1, 0, \dots, 0)$ of l and μ a partition of length k and maximal element $\leq n-k$. Then $\sigma_\lambda \smile \sigma_\mu = \sum_\nu \sigma_\nu$, where we sum over all partitions ν that can be obtained by adding one box to exactly l different rows of the Young diagram of μ.

Chern Classes of $\mathrm{N}(\iota)$

- Applying the Whitney product formula for characteristic classes, we obtain

$$c(\mathrm{N}(\iota)) = c(\mathrm{N}(\alpha_{n,k}))^{(n'-k')-(n-k)} \smile c(\mathrm{N}(\eta_{n,k}))^{k'-k}$$

- The only cohomology classes appearing in this products are $\sigma_{(1,\ldots,1,0,\ldots,0)}$ and $\sigma_{(l,0,\ldots,0)}$, hence the two Pieri formulas suffice for computing this product.
- For example, whenever $k, n-k \geq 2$ we have $(\sigma_1^1)^2 = \sigma_1^2 + \sigma_2^2$ and

$$c_1(\mathrm{N}(\iota)) = (n'-n)\sigma_1^1,$$

$$c_2(\mathrm{N}(\iota)) = (k'-k)\sigma_1^2 + ((n'-k')-(n-k))\sigma_2^2 + \binom{n'-n}{2}(\sigma_1^1)^2.$$

The Endomorphism Bundle

- For vector bundles $E, F \to X$ we define the bundle $\mathrm{Hom}(E, F)$ as $E^D \otimes F$. In particular, $\mathrm{End}\, E = \mathrm{Hom}(E, E)$.
- $\mathrm{TGr}(n, k) \simeq \mathrm{Hom}(\mathrm{N}(\alpha_{n,k})^D, \mathrm{N}(\eta_{n,k}))$
- Then
  $\mathrm{End}\, \mathrm{N}(\alpha_{n,k})^D \oplus \mathrm{TGr}(n, k) =$
  $\mathrm{Hom}(\mathrm{N}(\alpha_{n,k})^D, \mathrm{N}(\alpha_{n,k})^D) \oplus \mathrm{Hom}(\mathrm{N}(\alpha_{n,k})^D, \mathrm{N}(\eta_{n,k})) \simeq$
  $\mathrm{Hom}(\mathrm{N}(\alpha_{n,k})^D, \mathrm{N}(\alpha_{n,k})^D \oplus \mathrm{N}(\eta_{n,k})) \simeq$
  $\mathrm{Hom}(\mathrm{N}(\alpha_{n,k})^D, \mathbb{C}^n) = \mathrm{N}(\alpha_{n,k}) \otimes \mathbb{C}^n \simeq \bigoplus^n \mathrm{N}(\alpha_{n,k})$.
- The pullback of $\mathrm{N}(\alpha_{n,k})$ over the variety of complete flags in $\mathbb{C}^n$ splits into a direct sum of line bundles $L_1 \oplus \ldots \oplus L_k$.
- Hence the pullback of $\mathrm{End}\, \mathrm{N}(\alpha_{n,k})^D = \mathrm{End}\, \mathrm{N}(\alpha_{n,k})$ splits into $\mathbb{C}^k \oplus \bigoplus_{i \neq j} (L_i^D \otimes L_j)$.

The Endomorphism Bundle II

- $c(\mathrm{End}\,\mathrm{N}(\alpha_{n,k})) = \prod_{i\neq j}(1 - c_1(L_i) + c_1(L_j)) = \prod_{1\leq i<j\leq k}(1-(c_1(L_i) - c_1(L_j))^2)$
- The odd Chern classes of $\mathrm{End}\,\mathrm{N}(\alpha_{n,k})$ are zero and the even Chern classes are polynomials in Chern classes of $\mathrm{N}(\alpha_{n,k})$.
- Thus, we can compute the Chern classes of $\mathrm{TGr}(n,k)$ from Chern classes of $\mathrm{N}(\alpha_{n,k})$ only.
- $c_1(\mathrm{TGr}(n,k)) = c_1(\oplus^n \mathrm{N}(\alpha_{n,k})) = n\sigma_1^1$ and for $j > 2$:

$$c_j(\mathrm{TGr}(n,k)) =$$

$$c_j(\oplus^n \mathrm{N}(\alpha_{n,k})) - = \prod_{s=0}^{j-1} c_s(\mathrm{TGr}(n,k))c_{j-s}(\mathrm{End}\,\mathrm{N}(\alpha_{n,k})).$$

The case $k = 2$

- The formula is in particular nice for $c_j(\mathrm{TGr}(n,2))$.
- The only nonzero Chern class of $\mathrm{End}\,\mathrm{N}(\alpha_{n,k})$ is $c_2(\mathrm{End}\,\mathrm{N}(\alpha_{n,2})) = -(c_1(L_1) - c_1(L_2))^2 = -(c_1(L_1) + c_1(L_2))^2 + 4c_1(L_1)c_1(L_2) = -c_1(\mathrm{N}(\alpha_{n,2}))^2 + 4c_2(\mathrm{N}(\alpha_{n,2})) = -(\sigma_1^1)^2 + 4\sigma_1^2 = 3\sigma_1^2 - \sigma_2^2$.
- So we have $c_1(\mathrm{TGr}(n,2)) = n\sigma_1^1$ and for $j > 2$:

$$c_j(\mathrm{TGr}(n,2)) = c_j(\oplus^n \mathrm{N}(\alpha_{n,2})) - c_{j-2}(\mathrm{TGr}(n,2))c_2(\mathrm{End}\,\mathrm{N}(\alpha_{n,2})).$$