GEOMETRY THROUGH THE LENS OF DG MANIFOLDS

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OUTLINE

- The category of dg manifolds
- Examples of dg manifolds
- Examples of dg submanifolds
- BV-type correspondence
- Formal dg bundles

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THE CATEGORY OF DG MANIFOLDS

A dg manifold (a Q-manifold, A. Schwarz) is a $\mathbb{Z}$ ($\mathbb{Z}_2$ or $\mathbb{N}$) graded supermanifold endowed with a homological degree 1 vector field (called a Q-field or a Q-structure).

The category of dg-manifolds dgMan consists of:

- dg manifolds (objects);
- *dg* morphisms (or $Q-$morphisms) between two dg manifolds (M_1, Q_1) and (M_2, Q_2) - degree preserving maps $\phi\colon M_1 \to M_2$ with the vanishing field strength

$$F := Q_1\phi^* - \phi^* Q_2 = 0$$

THE CATEGORY OF DG MANIFOLDS

Given two *dg* manifolds, their graded super product is again a *dg* manifold with the product *dg* structure; one can show that dgMan satisfies the properties of a tensor category.

A dg submanifold is a graded immersed super submanifold such that the corresponding immersion is a $Q-$morphism.

The internal homomorphisms $\underline{Hom}(-,-)$ (or the super space of maps $M_1 \to M_2$) - in good cases is a new, possibly infinite-dimensional dg manifold (G.Bonavolonta, A. K., The smooth structure on $\underline{Hom}(-,-)$)

EXAMPLES OF DG MANIFOLDS

- A Lie algebroid $A \to X$. The Q-field on $A[1]$ is given by the Lichnerowicz differential
- Lie algebroids are in one-to-one correspondence with dg manifolds of degree 1 (A. Vaintrob)
- Furthermore, Lie algebroid morphisms are in one-to-one correspondence with $Q-$morphisms of these dg manifolds
- In particular, a Lie algebra $\mathfrak{g}$. The Lichnerowicz differential becomes the Chevalley-Eilenberg differential
- $T[1]M$ for a graded supermanifold; the Q-field is the de Rham operator
- Lie-infinity algebroids (non-negatively or $\mathbb{N}-$graded dg manifolds)

EXAMPLES OF DG MANIFOLDS: $L_\infty-$ALGEBRAS

Let L be a $\mathbb{Z}-$graded vector space and let $V = L[1]$.

Let Q be a formal homological vector field of degree 1 on V that vanishes at the origin.

We call (V, Q) a formal pointed Q-manifold.

The Taylor expansion of Q gives us a set of degree 1 linear maps

$$S^k V \to V[1]$$

for each $k \geq 0$.

EXAMPLES OF DG MANIFOLDS: L_∞−ALGEBRAS

Using the canonical isomorphisms of the graded vector spaces

$$S^k V \simeq \left(\Lambda^k L\right)[k]$$

we obtain a family of skew-symmetric maps

$$\Lambda^k L \to L[2-k]$$

for all $k \geq 0$, which satisfies the generalized Jacobi identities

L_∞−morphisms $L_1 \to L_2$ between L_∞−algebras are in one-to-one correspondence with Q−morphisms of the associated Q-manifolds

$$\left(L_1[1], Q_1\right) \to \left(L_2[1], Q_2\right)$$

EXAMPLES OF DG MANIFOLDS

- Symplectic dg manifolds (graded super symplectic manifolds whose symplectic structure is invariant under Q)
- In particular, the symplectic degree 2 dg manifold corresponding to a Courant algebroid (D. Roytenberg)
- The group-like objects in the category of dg manifolds are dg or Q-groups (B. Jubin, A.K., N. Poncin, V. Salnikov)
- The differential graded resolution of a (possibly) singular variety, an example of a non-positively graded dg manifold (A. Hancharuk, C. Laurent-Gengoux, T. Strobl, Koszul-Tate resolutions and decorated trees)

EXAMPLES OF DG MANIFOLDS: DG BUNDLES

A *dg* or $Q-$ bundle (A.K, T.Strobl) is a fibered bundle in the category dgMan, that is, a locally trivial $\mathbb{Z}-$graded bundle $\pi\colon E \to M$ over a *dg* manifold M, supplied with a total $Q-$structure, such that the projection map is a $Q-$morphism.

A *dg* section is a $Q-$morphism $\sigma\colon M \to E$, such that $\pi \circ \sigma = \mathrm{Id}$.

EXAMPLE OF A DG BUNDLE

Let $\pi_X\colon E \to X$ be a fibered bundle over a smooth manifold, then $\pi = \mathrm{d}\pi_X\colon (T[1]E, \mathrm{d}_E) \to (T[1]X, \mathrm{d}_X)$ is a $Q-$bundle. The tangent map to any section of π_X gives us a Q-section of π.

EXAMPLES OF DG MANIFOLDS: DG VECTOR BUNDLES

A dg bundle $\pi\colon E \to M$ is a *dg* vector bundle if E is a vector bundle over M and the $Q-$field on the total space of E is fiber-wise linear. Equivalently, Q commutes with the homogeneity structure which determines the structure of a vector bundle (J. Grabowski, M. Rotkiewicz).

EXAMPLES OF DG VECTOR BUNDLES

- For any dg manifold (M, Q), $T[k]M$ and $T^*[k]M$ are dg vector bundles over M for any k, where the $Q-$field on the total space is given by the Lie derivation along Q
- Any natural vector bundle over M with a Q-structure obtained in the same way
- Any chain complex is a dg vector bundle over a point

Let $\phi\colon M_1 \to M_2$ be a degree-preserving map between two dg manifolds, not necessarily a Q-morphism. Notice that $T[1]M_2$ is again a dg manifold with $Q_{tot} = \mathrm{d} + L_Q$, where d is the de Rham differential and L_Q is the super Lie derivation along Q.

Let us define a new degree preserving map $\tilde{\phi}\colon M_1 \to T[1]M_2$, called the generalized Cartan map as follows:

$$\tilde{\phi}^*(f\,\mathrm{d}h) = \phi^*(f)\ F(h)$$

The map $\tilde{\phi}\colon M_1 \to T[1]M_2$ is a Q-morphism (A.K., T.Strobl)

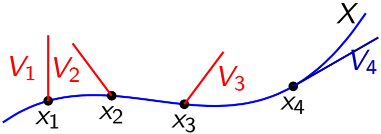
The generalized Cartan map is used to construct characteristic classes, as well as an alternative formulaic representation of the action functional of the AKSZ sigma model.

EXAMPLES OF DG SUBMANIFOLDS

A graded submanifold of dimension (m, m) of $T[1]Y$ will consists of the following data:

- a smooth submanifold $X \subset Y$ of dimension m
- a vector subbundle $V \subset TY_{|X}$ of rank m, such that $V_x \subset T_x Y$ is an $m-$dimensional vector subspace for each $x \in X$

Some of the subspaces may coincide with the tangent space to the submanifold X at the corresponding point, but this is not the general case



There is one-to-one correspondence between:

1. smooth maps between manifolds $Y_1 \to Y_2$ and $Q-$morphisms $T[1]Y_1 \to T[1]Y_2$;
2. $m-$dimensional submanifolds $X \subset Y$ and graded $Q-$submanifolds of $T[1]Y$ of graded real dimension (m, m).

The latter is given by the inclusion $T[1]X \hookrightarrow T[1]Y$.

The proof of both statements is straightforward.

In particular, for the second statement, consider the ideal $I_X \subset C^\infty(Y)$ of smooth functions vanishing on X. Then, since (X, V) defines a dg submanifold, $\mathrm{d}I_X$ must annihilate V. For dimensional reasons, $V = \left(\mathrm{d}I_X\right)^\perp_X$, therefore $V = TX$.

Let Y be a complex manifold; we associate to it a non-negatively graded complex dg manifold $\mathcal{M} = T^{0,1}[1]Y$, on which the algebra of functions is $\Omega^{0,\bullet}(Y) = \Gamma_Y\left(\wedge\left(T^{0,1}Y\right)^{\vee}\right)$ and the $Q-$structure is the Dolbeault differential $\bar{\partial}$.

THEOREM

There is one-to-one correspondence between:

1. holomorphic maps between complex manifolds $Y_1 \to Y_2$ and $Q-$morphisms $T^{0,1}[1]Y_1 \to T^{0,1}[1]Y_2$;
2. complex $k-$dimensional submanifolds $X \subset Y$ and graded $Q-$submanifolds of $T^{0,1}[1]Y$ of graded real dimension $(2k, 2k)$.

The latter is given by the inclusion $T^{0,1}[1]X \hookrightarrow T^{0,1}[1]Y$.

The proof of the second statement of this theorem is somewhat more complicated than one might expect.

Consider a Lie algebra $\mathfrak{g}$ and the associated dg manifold $\mathfrak{g}[1]$ where the Q-field is determined by the Chevalley-Eilenberg differential.

Assume thay $\mathfrak{g}$ is finite-dimensional with a basis $\{e_i\}_{i=1}^{\dim \mathfrak{g}}$ such that

$$[e_i, e_j] = \sum_k C_{ij}^k e_k$$

Let x^i be the corresponding linear degree 1 coordinates on $\mathfrak{g}$. Then

$$Qx^k = -\frac{1}{2} \sum_{i,j} C_{ij}^k x^i x^j$$

for $1 \leq i, j, k \leq \dim \mathfrak{g}$

Q-submanifolds of $\mathfrak{g}[1]$ are in one-to-one correspondence with Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra and α be a complementary vector subspace such that

$$\mathfrak{g} = \mathfrak{h} \oplus \alpha$$

Deformations of $\mathfrak{h}$ as vector subspaces are determined by graphs of linear maps $\phi \colon \mathfrak{h} \to \alpha$

Under what requirements on a linear map ϕ does the graph of ϕ define a Lie subalgebra?

In other words, the following condition is satisfied

$$\left[v_1 + \phi(v_1), v_2 + \phi(v_2)\right] \in \mathit{graph}(\phi) = \{v + \phi(v) \,|\, v \in \mathfrak{g}\}$$

for all $v_1, v_2 \in \mathfrak{h}$

Denote the projections of an element of $\mathfrak{g}$ onto $\mathfrak{h}$ and α by the subscripts 1 and 2, respectively. Then

$$\left[v_1 + \phi(v_1), v_2 + \phi(v_2)\right] = v + \phi(v) + B$$

where

$$v = [v_1, v_2]_1 + \left[v_1, \phi(v_2)\right]_1 - \left[v_2, \phi(v_1)\right]_1 + \left[\phi(v_1), \phi(v_2)\right]_1 \in \mathfrak{h}$$

$$B = \big[v_1, \phi(v_2)\big]_2 - \big[v_2, \phi(v_1)\big]_2 - \phi\big(\big[v_1, v_2\big]_1\big) + \big[\phi(v_1), \phi(v_2)\big]_2$$

$$+\phi\Big(- \big[v_1, \phi(v_2)\big]_1 + \big[v_2, \phi(v_1)\big]_1\Big)$$

$$-\phi\Big(\big[\phi(v_1), \phi(v_2)\big]_1\Big) \in \alpha$$

- The graph of ϕ determines a Lie subalgebra if and only if

$$B = 0$$

- In general, this equation is cubic with respect to ϕ

$$B = \big[v_1, \phi(v_2)\big]_2 - \big[v_2, \phi(v_1)\big]_2 - \phi\big(\big[v_1, v_2\big]_1\big) + \big[\phi(v_1), \phi(v_2)\big]_2$$

$$+\phi\Big(- \big[v_1, \phi(v_2)\big]_1 + \big[v_2, \phi(v_1)\big]_1\Big)$$

$$-\phi\Big(\big[\phi(v_1), \phi(v_2)\big]_1\Big)$$

- The red (cubic) part is equal to zero if α is a Lie subalgebra.
- The blue part disappears if α is an ideal, i.e. the projection $\mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism
- In addition, the green part vanishes if α is an Abelian ideal

BV-TYPE CORRESPONDENCE

Dg interpretation of the space of Lie subalgebras

Consider the graded space $\mathcal{S}_m(\mathfrak{g})$ of submanifolds of $\mathfrak{g}[1]$ of graded dimension $(0, m)$.

- The zero degree part of $\mathcal{S}_m(\mathfrak{g})$ coinsides with the Grassmanian manifold $Gr_m(\mathfrak{g})$ of $m-$dimensional vector subspaces of $\mathfrak{g}$
- $\mathcal{S}_m(\mathfrak{g})$ admits a (non-canonical) structure of a graded vector bundle over $Gr_m(\mathfrak{g})$, isomorphic to $\bigoplus_{j\neq 1} \Lambda^j \eta^* \otimes \lambda$, where η is the canonical rank m bundle over $Gr_m(\mathfrak{g})$ and $\lambda = \mathfrak{g}/\eta$
- The Chevalley-Eilenberg Q-structure on $\mathfrak{g}[1]$ generates a Q-structure $\tilde{Q}$ on $\mathcal{S}_m(\mathfrak{g})$ such that the Lie subalgebras of $\mathfrak{g}$ of dimension m will correspond to points of the zero locus of $\tilde{Q}$

Let (M, Q) be a dg manifold and let z be a point of the zero locus of Q, i.e.

$$Q_z = 0$$

Consider a formal neighborhood U of z.

Suppose we have found a way to identify U with a formal neighborhood of zero in $V = T_z M$.

- Thus, we obtain a formal pointed Q-manifold $\big(V, Q\big)$, where the Q-structure on V is induced by the Q-structure on M.
- Using the correspondence between L_∞ algebras and formal pointed $Q-$manifolds we also obtain the infinity algebra L, where $L = V[-1]$
- Deligne principle Formal deformations inside the zero locus Q are in one-to-one correspondence with the solutions of the (generalized) Maurer-Cartan equation

Solutions to equations ↔ Zero locus of a $Q-$field

Deformations of solutions ↔ Maurer-Cartan equations

Infinitesimal symmetries ↔ Infinitesimal gauge symmetries of the form $[Q, \epsilon]$, where ϵ is a vector field of degree -1

Equivalence of equations ↔ Equivalence of the corresponding dg manifolds

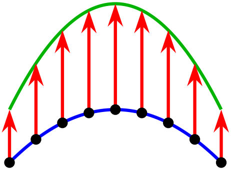
The most interesting examples of this construction are obtained in the case of spaces of geometric structures associated with Q-manifolds, which are themselves Q-manifolds (or Q-spaces, if it is not a smooth graded supermanifold)

For example, it can be a graded superspace of all submanifolds of fixed superdimension in a Q-manifold. This graded superspace is itself endowed with a canonical $Q-$structure.

Let N be a $Q-$submanifold of a $Q-$manifold M. It is regarded as a point z of the zero locus of the $Q-$structure on the graded super space of all submanifolds of M.

Suppose that the manifold Y is a bundle over X, $\pi\colon Y \to X$ and accordingly X is a section of this bundle. Then the tangent space at the point z can be identified with the space of $\pi-$vertical vector fields over X.

Moreover, by choosing a $\pi-$fiberwise linear connection, we identify a local deformation of X with a $\pi-$vertical vector field over X.



In the case of dg manifolds, the structure of a bundle $\pi\colon M \to N$ together with the $\pi-$fiberwise connection provides the space of vertical vector fields (with shifted degree) with the structure of an algebra L_∞

BV-TYPE CORRESPONDENCE: EXAMPLES

- For any Q-manifold, an L_∞ algebra on the space of vector fields on M (the Kapranov structure, S. Seol, M. Stiénon, P. Xu) is constructed. To do this, consider the diagonal $M \subset M \times M$ as a Q-submanifold in $M \times M$. The tangent space to this submanifold considered as a point in the space of all submanifolds is identified with sections of TM. The formal neighborhood of this point is identified with the tangent space by choosing a linear connection in TM.

- The space of Lagrangian submanifolds of a symplectic $Q-$manifold. For a Lagrangian Q-submanifold N we obtain a P_∞ structure on the space of functions on N (M. Cueca, J. Schnitzer; also A.K., V. Rubtsov, V. Salnikov, unpubl.)

The structure of a global bundle of M over N is a too strong condition. To describe local or formal deformations of N, the structure of a local or formal bundle, respectively, is sufficient. This means that a local or formal neighborhood of N in M is a bundle over N.

By analogy with deformations of Lie subalgebras in a Lie algebra, the following options (global, local or formal) of the bundle π are possible, listed in order of increasing strength:

- The fibers of π are $Q-$submanifolds
- The bundle π is a $Q-$bundle
- The bundle is isomorphic to the normal bundle, i.e. the $Q-$structure is linearizable around N

In a holomorphic context, the above conditions are equivalent to the following, respectively

- The fibers of the bundle are holomorphic submanifolds (example: twistor spaces)
- The bundle is holomorphic
- The holomorphic tubular neighborhood theorem holds

Theorem

For any holomorphic submanifold X of a complex manifold Y there exists a projection of the formal neighborhood of X onto X whose fibers are holomorphic.

The proof is based on the observation that the $k+1$-jet space of holomorphic submanifolds is an affine bundle over the k-jet space.

FORMAL DG BUNDLES

Let $N \subset M$ b a $Q-$submanifold and $J = I_N$ be the ideal of functions vanishing on N.

Denote by $\mathcal{A}$ and $\mathcal{A}^{(k)}$ the algebras of functions on M and $N^{(k)}$, the kth neighborhood of N, respectively.

In particular, $\mathcal{A}^{(0)}$ is isomorphic to the algebra of functions on N

One has the following isomorphisms

$$\mathcal{A}^{(k)} \simeq \mathcal{A}/J^{k+1}$$

and

$$J^k/J^{k+1} \simeq \Gamma(S^k\nu^*)$$

where ν is the normal bundle, $\nu = TM_{|N}/TN$.

- The restriction map $\mathcal{A}^{(k+1)} \to \mathcal{A}^{(k)}$ extends to the following short exact sequence

$$0 \to \Gamma(S^k \nu^*) \to \mathcal{A}^{(k+1)} \to \mathcal{A}^{(k)} \to 0$$

- A splitting $\Phi^{(k)}$ of $\mathcal{A}^{(k)} \to \mathcal{A}^{(0)}$ is equivalent to the choice of a bundle structure $N^{(k)} \to N$
- Let $\Phi_1^{(k+1)}$ and $\Phi_2^{(k+1)}$ be two splitting of $\mathcal{A}^{(k+1)} \to \mathcal{A}^{(0)}$ that coincide modulo J^k. Then

$$\Delta\Phi^{(k+1)} := \Phi_1^{(k+1)} - \Phi_2^{(k+1)}$$

can be identified with a degree 0 section of $TN \otimes S^k \nu^*$

- $TN \otimes S^k\nu^*$ has a canonical structure of a dg vector bundle over N. Thus sections of $TN \otimes S^k\nu^*$ is a chain complex.
- Assume that $\Phi^{(k)} \colon \mathcal{A}^{(0)} \to \mathcal{A}^{(k)}$ is a chain map. This is equivalent to that the corresponding bundle $N^{(k)} \to N$ is a dg bundle. Choose any prolongation $\Phi^{(k+1)}$, which is not a chain map, in general. More precisely, there is the curvature $\kappa_{\Phi^{(k+1)}}$, identified with a degree 1 section of $TN \otimes S^k\nu^*$, which vanishes if an only if $\Phi^{(k+1)}$ is a chain map.

THEOREM

- The curvature $\kappa_{\Phi^{(k+1)}}$ is a cocycle
- Its cohomology class does not depend on the choice of the prolongation $\Phi^{(k+1)}$

Thank you for your attention!