# On nearly parallel $G_2$-manifolds

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# $G_2$-structures on 7-manifolds

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Fernández and Gray (1982) worked out a classification of $G_2$-structures considering components of the intrinsic torsion; these can be determined from $d\varphi$ and $*_\varphi\varphi$.

# Nearly parallel $G_2$-structures

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- $(M, g)$ is *Sasaki–Einstein* if $\mathrm{Hol}(g_{\mathrm{cone}}) = SU(4)$ (a Ricci-flat Kähler metric, but not hyper-Kähler)

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- $(M, g)$ is *proper* if $\text{Hol}(g_{\text{cone}}) = \text{Spin}(7)$ (a Ricci-flat but not Kähler metric)

# $G_2$-structures on Sasaki–Einstein manifolds

A Riemannian $2n+1$-manifold $(S, g)$ is called *Sasakian* when the cone metric $g_{\text{cone}} = dr^2 + r^2 g$ on $\mathbb{R}_{>0} \times S$ is Kähler for some (integrable) complex structure $J$.

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A compact Sasakian–Einstein manifold $(S, g)$ is called *regular* if $J\partial_r$ integrates to a free $S^1$-action on $S$. Then $S$ is a principal $S^1$-bundle $S^1 \to S \xrightarrow{p} X$ over a Kähler–Einstein manifold $X$ with positive curvature and with the Euler class $c_1 = [\omega_X]$ given by the Kähler form ($X$ is then a simply-connected, smooth **Fano variety**)

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A regular Sasaki–Einstein 7-manifold $(S, g)$ has a canonical *$S^1$-family of nearly parallel $G_2$-structures* inducing the metric $g$

$$\varphi_t = \Omega \wedge \eta + \mathrm{Re}(e^{-it}\Psi),$$

where $\Omega = p^* \omega_X$, $\Psi = \partial_r \lrcorner \widehat{\Psi}|_{r=1}$, $\widehat{\Psi}$ a unit length holomorphic 4-form on the cone $\mathbb{R}_{>0} \times S$ (Alexandrov & Semmelmann,2012)

# Associative 3-folds

Let $M$ be a 7-manifold with a $G_2$-structure $\varphi$. An oriented 3-dimensional submanifold $Y \subset M$ is called *associative* if

$$\varphi|_Y = \mathrm{vol}_Y$$

where $\mathrm{vol}_Y$ is the volume form of the metric induced by $g(\varphi)$. (In general, one only has $\varphi|_Y \leq \mathrm{vol}_Y$.) If $d\varphi = 0$, then an associative $Y$ is an instance of a **calibrated submanifold** in the sense of Harvey and Lawson; if $Y$ is also compact then $Y$ is volume-minimizing in its homology class and thus minimal.

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If instead $\varphi$ is a nearly parallel, then $Y \times \mathbb{R}_+$ is a calibrated 4-submanifold for the torsion-free Spin(7)-structure on the Riemannian cone $M \times \mathbb{R}_+$ and $Y$ is still **minimal**.

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If instead $\varphi$ is a nearly parallel, then $Y \times \mathbb{R}_+$ is a calibrated 4-submanifold for the torsion-free Spin(7)-structure on the Riemannian cone $M \times \mathbb{R}_+$ and $Y$ is still **minimal**. Moreover, in the Fernández-Gray classification of $G_2$-structures, the latter two possibilities give the largest class where every associative 3-fold is minimal (Ball and Madnick, 2021).

# Associative 3-folds in Sasaki–Einstein 7-manifolds I

When a (nearly parallel) $G_2$-structure on a 7-manifold $S$ comes from a Sasaki–Einstein structure, the associative 3-folds $Y \subset S$ can be:

- *'invariant'* submanifolds or
- *special Legendrian* submanifolds,

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## Proposition

*Let $S$ be a regular Sasaki–Einstein 7-manifold with contact form $\eta$, thus a principal $S^1$-bundle $\pi : S \to X$ over a Kähler–Einstein projective 3-fold $X$ with Kähler form $\omega$ and $d\eta = \pi^*\omega$. Let $\varphi_t$ be the induced $S^1$-family of nearly parallel $G_2$ 3-forms.*

*Then, given a complex curve $\Sigma$ in $X$, the $Y_\Sigma = \pi^{-1}(\Sigma) \subset S$ is an invariant minimal associative with respect to $\varphi_t$ for each $t$.*

# Associative 3-folds in Sasaki–Einstein 7-manifolds II

A minimal associative $Y_\Sigma$ is invariant under isometric $S^1$-action on the principal bundle $S$ and every deformation of the complex curve $\Sigma$ in $X$ induces an associative deformation of $Y_\Sigma$.

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Examples arising from the above Proposition are:

- the $S^1$-bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ with $c_1 = [\omega]$, $\omega = \omega_1 + \omega_2 + \omega_3$ where $\omega_k$ is a Kähler form on the $k$-th factor generating $H^2(\mathbb{C}P^1, \mathbb{Z})$. Equivalently, $S = Q(1,1,1) = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$;
- the $S^1$-bundle over $\mathbb{C}P^1 \times P_k$ with $P_k$ a del Pezzo surface, $(3 \le k \le 8)$,$c_1 = [\omega]$, $\omega = \omega_1 + \omega_P$ with $[\omega_P] \in H^2(P_k, \mathbb{Z})$

Taking $\Sigma = \mathbb{C}P^1 \times (point)$ we obtain a minimal associative 3-sphere $Y_\Sigma \cong S^3$.

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More generally one can take $\Sigma$ to be the graph of a holomorphic embedding $\mathbb{C}P^1 \to P$ or $\mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$. In the latter case the ambiguity corresponds to a generic choice of two rational functions of one complex variable.

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*Let $S$ be a regular Sasaki–Einstein 7-manifold with contact form $\eta$ arising from a principal $S^1$-bundle $\pi : S \to X$ with Euler class $c_1 = [\omega]$, where $X$ is a Kähler–Einstein Fano 3-fold with Kähler form $\omega$ and $d\eta = \pi^*(\omega)$. Let $\varphi_t$ be the corresponding 1-parameter family of induced nearly parallel $G_2$ forms.*

*Then (i) for each compact special Legendrian submanifold $Y \subset S$, the restriction $\pi|_Y : Y \to Y_X$ is a finite covering of a Lagrangian submanifold $Y_X \subset X$.*

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*(ii) If $Y_X \subset X$ is a compact simply-connected Lagrangian submanifold, thus a Lagrangian 3-sphere, then $Y_X$ lifts to an $S^1$-family of Legendrian submanifolds $Y_s \subset S$ such that $\pi(Y_s) = Y_X$ for each $s \in S^1$.*

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*(iii) Assume that $\tau : X \to X$ is an isometric anti-holomorphic involution. If the fixed point set $Y_X \subset X$ of $\tau$ is non-empty, then $Y_X$ is Lagrangian and diffeomorphically lifts to a special Legendrian (hence minimal associative) submanifold of $S$.*

# Examples of Legendrian associatives

We consider again the 7-manifold $Q(1,1,1)$ and the $S^1$-bundle over a product of $\mathbb{C}P^1$ and the del Pezzo surface $P_3$.

## Proposition

*Let $\pi_M : Q(1,1,1) \to X = S^2 \times S^2 \times S^2$ be the principal $S^1$-bundle and $\varphi_t$ the 1-dimensional family of nearly parallel $G_2$-structures on $Q(1,1,1)$. Let $L \subset X$ be a 3-torus defined by $\theta_j = \pi/2$, $j = 1,2,3$, in the spherical coordinates $\phi_j, \theta_j$ on $X$.*

*Then $L$ lifts via $\pi_M$ to a family of minimal Legendrian 3-tori $L_s \subset M$, $s \in \mathbb{R}/2\pi\mathbb{Z}$. For each $s$, the 3-torus $L_s$ is associative with respect to $\varphi_t$ for all $t$.*

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For the next result it is important that $P_3$ is a toric variety and has a Kähler–Einstein metric invariant under the $\mathbb{C}^*$ action.

## Proposition

*There exists a (minimal) associative 3-torus in the nearly parallel $G_2$-manifold $(S_3, \varphi_t)$, where $S_3$ is the $S^1$-bundle over $\mathbb{C}P^1 \times P_3$ (with $c_1$ given by the Kähler form as before).*

# Associative 3-folds in Aloff–Wallach spaces

The Aloff–Wallach spaces are compact simply-connected 7-manifolds defined as the quotients $W_{k,l} = SU(3)/S^1_{k,l}$ of $SU(3)$ by a circle subgroup $S^1_{k,l} = \mathrm{diag}(e^{ik\theta}, e^{il\theta}, e^{im\theta})$, where $k > 0$, $l \neq 0$ are coprime integers and $k + l + m = 0$.

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By considering a basis of left-invariant 1-forms on $SU(3)$, one can write down a family of homogeneous co-closed $G_2$ 3-forms $\varphi_W$ depending on 4 real parameters. If $(k, l) \neq (1, \pm 1)$, then up to homotheties exactly 2, of these $\varphi_W$ are (proper) nearly parallel (Cabrera, Monar, Swann 1996).

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If $(k,l) \neq (1,-1)$, then there is a fibre bundle

$$\pi_{k,l} : W_{k,l} = SU(3)/S^1_{k,l} \to SU(3)/U(2) \cong \mathbb{C}P^2$$

with fibres $S^3/\mathbb{Z}_{|k+l|}$, corresponding to an embedding of $U(2)$ as a subgroup consisting of the block-diagonal matrices with blocks $Ae^{i\theta}$ and $e^{-2i\theta}$, $A \in SU(2)$.

This fibration $\pi_{k,l}$ is not unique. The Weyl group of $SU(3)$ contains an element of order 3 which induces a diffeomorphism $\upsilon : W_{k,l} \to W_{l,m}$. The composition $\pi_{l,m} \circ \upsilon$ defines a different fibration, in general by different spherical space forms.

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### Theorem

*Let $\varphi_W$ be a homogeneous (left-invariant) nearly parallel $G_2$-structure on the Aloff–Wallach space $W_{k,l}$, with $(k,l) \neq (1,\pm 1)$. Then the fibres of $\pi_{k,l}$ are embedded minimal associative 3-folds with respect to $\varphi_W$.*
*Furthermore, for suitably 'generic' $k,l$, the Aloff–Wallach space $W_{k,l}$ has three different 4-dimensional deformation families of minimal associative spherical space forms.*

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On $W_{1,-1}$ there is only one homogeneous nearly parallel $G_2$ structure. and the fibres of $\pi_{1,-1}$ are minimal associative $S^2 \times S^1$'s. On $W_{1,1}$ we only show associative $S^3/\mathbb{Z}_2$ for one of the two homogeneous nearly parallel $G_2$ structure which is 3-Sasakian. On the other hand, Ball and Madnick (2022) constructed in $W_{1,1}$ associative 3-folds diffeomorphic to $S^1$-bundles over a compact surface of any genus $g \geq 0$.

# Some open questions

- Our theorem on minimal associatives of Legendrian type Sasaki–Einstein 7-manifolds can probably produce more examples. One would need to determine which Kähler–Einstein Fano 3-folds, e.g. (del Pezzo surfaces)$\times\mathbb{C}P^1$, have an isometric antiholomorphic involution. One challenge is that the Kähler–Einstein metrics are often given implicitly via existence results for PDEs and vanishing of certain 'obstruction' invariants.
- It would be interesting to find a more systematic description of minimal associative 3-folds in the Aloff–Wallach which includes the known examples.