On some new constructions in quantum vertex algebra theory

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Quantum Yang–Baxter equation

- first appeared in statistical mechanics (the 8-vertex model) in late 1970s

Definition

Let V be a vector space and R a linear map $V \otimes V \to V \otimes V$. The identity of operators on $V \otimes V \otimes V$,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{YBE}$$

is called quantum Yang–Baxter equation and the map R is called R-matrix. The indices indicate the tensor factors of $V \otimes V \otimes V$ so that, e.g., $R_{12} = R \otimes 1$.

Example

The identity and the permutation operator on $\mathbb{C}^N \otimes \mathbb{C}^N$,

$$I = \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} \quad \text{and} \quad P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}$$

are solutions of (YBE).

Examples of rational and trigonometric R-matrices

The Yang–Baxter equation (YBE) is often generalized as follows:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \qquad \text{(a-YBE)}$$

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x). \qquad \text{(m-YBE)}$$

Example

The Yang R-matrix is a solution of (a-YBE),

$$R(u) = I - u^{-1}P.$$

Example

The trigonometric R-matrix is a solution of (m-YBE),

$$R^q(x) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \frac{1-x}{q-q^{-1}x} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} e_{ii} \otimes e_{jj}$$

$$+ \frac{\left(q-q^{-1}\right)x}{q-q^{-1}x} \sum_{\substack{i,j=1 \\ i>j}}^{N} e_{ij} \otimes e_{ji} + \frac{q-q^{-1}}{q-q^{-1}x} \sum_{\substack{i,j=1 \\ i<j}}^{N} e_{ij} \otimes e_{ji}.$$

Example

The elliptic R-matrix of the eight-vertex model is also a solution of (m-YBE),

$$R(z) = \begin{pmatrix} a(z) & 0 & 0 & d(z) \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ d(z) & 0 & 0 & a(z) \end{pmatrix}$$

w.r.t. the basis $(e_1 \otimes e_1,\, e_1 \otimes e_2,\, e_2 \otimes e_1,\, e_1 \otimes e_2)$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$, where $e_1 = (1,0)$, $e_2 = (0,1)$.

Its matrix entries, where $a(z)$ and $b(z)$ (resp. $c(z)$ and $d(z)$) possess only even (resp. odd) powers of z, are uniquely determined by

$$a(z) + d(z) = q^{-1/2} f(z^2)^{-1} \frac{\alpha(z^{-1})}{\alpha(z)} \quad \text{and} \quad b(z) + c(z) = q^{1/2} f(z^2)^{-1} \frac{1+q^{-1}z}{1+qz} \frac{\beta(z^{-1})}{\beta(z)},$$

where

$$\alpha(z) = \frac{(p^{1/2}qz;p)_\infty}{(p^{1/2}q^{-1}z;p)_\infty} \prod_{k\geqslant 1} f(p^k z^2), \qquad \beta(z) = \frac{(-pqz;p)_\infty}{(-pq^{-1}z;p)_\infty} \prod_{k\geqslant 1} f(p^k z^2),$$

$$f(z) = \frac{(z;q^4)_\infty (zq^4;q^4)_\infty}{(zq^2;q^4)_\infty^2}, \qquad (z;p)_\infty = \prod_{k\geqslant 0} (1 - zp^k).$$

Quantum groups

- first examples appeared in the works of L. Fadeev, P. Kulish, N. Reshetikhin in the context of quantum integrable systems in early 1980s
- certain wide class of Hopf algebras introduced by V. Drinfeld and M. Jimbo around 1985, which are deformations of the universal enveloping algebras of certain Lie algebras $\rightsquigarrow$ Drinfeld–Jimbo-type quantum groups
- links with many areas of mathematics (Lie groups, Lie algebras and their representations, knot theory, noncommutative geometry etc.) and physics (quantum inverse scattering method, elementary particle physics, conformal and quantum field theories)

Remark

- Suppose V is a module for some quantum group U_q. Then $V \otimes V$ is equipped by the U_q-module structure due to the underlying Hopf algebra structure.
- There exists a distinct element $\mathfrak{R} \in U_q \widetilde{\otimes} U_q$, the universal R-matrix satisfying the (YBE), i.e. such that $\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12}$.
- The action of $\mathfrak{R}$ over $V \otimes V$ produces a solution of (YBE).

Affine Lie algebras

- $\mathfrak{g}$ ... a simple Lie algebra, e.g., set

$$\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ a + d = 0 \right\}$$

- $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$ ... affine Lie algebra:

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\langle x, y\rangle \delta_{m+n\,0} C, \qquad [x \otimes t^m, C] = 0$$

  for all $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$
- $\widehat{\mathfrak{g}} \subset U(\widehat{\mathfrak{g}})$ ... the universal enveloping algebra of $\widehat{\mathfrak{g}}$

Definition

Let $c \in \mathbb{C}$. The vacuum $\widehat{\mathfrak{g}}$-module $V^c(\mathfrak{g})$ of level c is defined by

$$V^c(\mathfrak{g}) = U(\widehat{\mathfrak{g}})/U(\widehat{\mathfrak{g}})\,\langle \mathfrak{g} \otimes \mathbb{C}[t], C - c \cdot 1\rangle\,.$$

Vertex algebras

- introduced by R. Borcherds in 1986
- they generalize the notions of commutative associative algebra and Lie algebra
- applications in many areas of mathematics (finite simple groups, infinite-dimensional Lie algebras, $\mathcal{W}$-algebras) and physics (two-dimensional conformal field theory, string theory)

Definition (A sketch)

A vertex algebra is a pair (V, Y), where V is a vector space and Y a linear map

$$Y = Y(\cdot, z)\colon V \to \operatorname{End} V[[z, z^{-1}]]$$
$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

which satisfies the locality, for any $a, b \in V$ there exists an integer $r \geqslant 0$ such that

$$(z_1 - z_2)^r Y(a, z_1) Y(b, z_2) = (z_1 - z_2)^r Y(b, z_2) Y(a, z_1),$$

the weak associativity, for any $a, b, c \in V$ there exists an integer $r \geqslant 0$ such that

$$(z_0 + z_2)^r Y(a, z_0 + z_2) Y(b, z_2) c = (z_0 + z_2)^r Y(Y(a, z_0) b, z_2) c,$$

and some other axioms. . .

Affine vertex algebras

Suppose $\mathfrak{g}$ is a simple Lie algebra and $h^\vee$ the dual Coxeter number for $\mathfrak{g}$.

Theorem (I. Frenkel, Y.-C. Zhu, 1992)

Let $c \in \mathbb{C}$. The vacuum module $V^c(\mathfrak{g})$ can be equipped with the structure of vertex algebra (universal affine vertex algebra) so that

(irreducible) restricted $\widehat{\mathfrak{g}}$-modules of level $c \in \mathbb{C}$	$=$	*(irreducible) modules for the vertex algebra $V^c(\mathfrak{g})$.*

Theorem (B. Feigin, E. Frenkel, 1992)

The center of the universal affine vertex algebra $V^{-h^\vee}(\mathfrak{g})$ at the critical level $c = -h^\vee$ (Feigin–Frenkel center) is an algebra of polynomials in infinitely many variables.

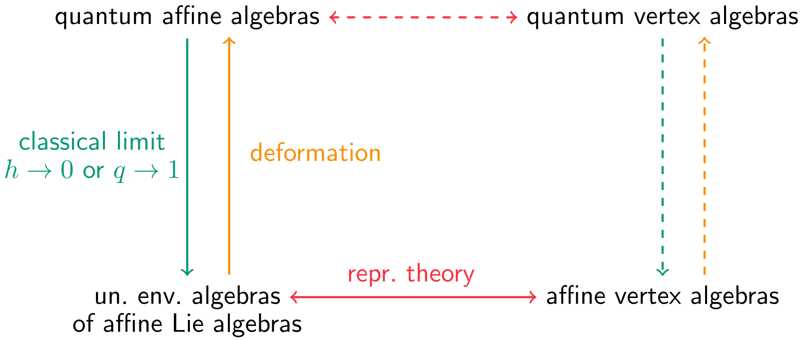
The explicit formulas for generators of the Feigin–Frenkel center were given by

- A. Chervov, D. Talalaev, 2006 and A. Chervov, A. Molev, 2009 for type A
- A. Molev, 2013 for types B, C and D
- A. Molev, E. Ragoucy, N. Rozhkovskaya, 2016 for type G_2.

Quantum vertex algebras?

Question (I. Frenkel, N. Jing)

Can we associate quantum vertex algebras to quantum affine algebras?



Early development of qVA theory:

- E. Frenkel, N. Reshetikhin, 1997 $\rightsquigarrow$ *deformed chiral algebras*
- P. Etingof, D. Kazhdan, 2000 $\rightsquigarrow$ *quantum vertex operator algebras*
- R. E. Borcherds, 2001 $\rightsquigarrow$ *quantum vertex algebras*
- B. Bakalov, V. G. Kac, 2003 $\rightsquigarrow$ *field algebras*
- H.-S. Li, 2003 $\rightsquigarrow$ *axiomatic* G_1*-vertex algebras / nonlocal vertex algebras*
- I. I. Anguelova, M. J. Bergvelt, 2009 $\rightsquigarrow$ H_D*-quantum vertex algebras*
- H.-S. Li, 2011 $\rightsquigarrow$ ϕ*-coordinated modules*

Some more recent results:

- A. De Sole, M. Gardini, V. G. Kac, 2020 $\rightsquigarrow$ structure theory of quantum VAs
- C. Boyallian, V. Meinardi, 2022 $\rightsquigarrow$ *quantum conformal algebras*
- N. Jing, F. Kong, H.-S. Li, S. Tan, 2021/22 $\rightsquigarrow$ *equivariant* ϕ*-coordinated modules*; deforming VAs by VBAs; quantum VAs for (twisted) qAAs
- F. Kong, 2023 $\rightsquigarrow$ quantum VAs for untwisted quantum affinization algebras
- E. Herscovich, 2023 $\rightsquigarrow$ *categorical quantum vertex algebras*
- F. Chen, X. Huang, F. Kong, S. Tan, 2024 $\rightsquigarrow$ qVAs for type A qTAs

(A sketch of) Etingof–Kazhdan's definition of qVA, 2000

A *quantum vertex algebra* is a triple $(\mathcal{V}, Y, \mathcal{S})$, where

- $\mathcal{V}$ is a topologically free $\mathbb{C}[[h]]$-module,
- $Y(\cdot, z)$ is a $\mathbb{C}[[h]]$-module map

$$\begin{aligned} Y(\cdot, z) \colon \mathcal{V} \otimes \mathcal{V} &\to \mathcal{V}[[z, z^{-1}]] \\ a \otimes b &\mapsto Y(a, z)b = \sum_{n \in \mathbb{Z}} a_n b z^{-n-1}, \end{aligned}$$

- $\mathcal{S} = \mathcal{S}(z) \colon \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}[[z, z^{-1}]]$ is a braiding, a $\mathbb{C}[[h]]$-module map which satisfies $\mathcal{S}\big|_{h=0} = 1$ and (a-YBE),

$$S_{12}(z_1) S_{13}(z_1 + z_2) S_{23}(z_2) = S_{23}(z_2) S_{13}(z_1 + z_2) S_{12}(z_1).$$

- The vertex operator map Y possesses the *$\mathcal{S}$-locality property*, for any $a, b \in \mathcal{V}$ and $n \geqslant 0$ there exists $r \geqslant 0$ such that for any $c \in \mathcal{V}$

$$\begin{aligned} &(z_1 - z_2)^r Y(z_1) \big(1 \otimes Y(z_2)\big) \big(\mathcal{S}(z_1 - z_2)(a \otimes b) \otimes c\big) \\ &\qquad - (z_1 - z_2)^r Y(b, z_2) Y(a, z_1) c = 0 \mod h^n. \end{aligned}$$

- some other axioms...

The trigonometric R-matrix $R^q(x)$ of type A

- The R-matrix over $\mathbb{C}(q)$ defined by

$$R^q(x) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \frac{1-x}{q-q^{-1}x} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} e_{ii} \otimes e_{jj}$$

$$+ \frac{\left(q-q^{-1}\right)x}{q-q^{-1}x} \sum_{\substack{i,j=1 \\ i>j}}^{N} e_{ij} \otimes e_{ji} + \frac{q-q^{-1}}{q-q^{-1}x} \sum_{\substack{i,j=1 \\ i<j}}^{N} e_{ij} \otimes e_{ji}.$$

- It satisfies the multiplicative Yang–Baxter equation (m-YBE),

$$R^q_{12}(x) R^q_{13}(xy) R^q_{23}(y) = R^q_{23}(y) R^q_{13}(xy) R^q_{12}(x).$$

Adapting the R-matrix: $\mathbb{C}(q) \rightsquigarrow \mathbb{C}[[h]]$

Set

$$R^h(x) = \left(R^q(x)\right)\big|_{q=e^h}.$$

- The R-matrix over $\mathbb{C}[[h]]$ defined by

$$R^h(x) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \frac{1-x}{e^h - e^{-h}x} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} e_{ii} \otimes e_{jj} + \frac{\left(e^h - e^{-h}\right)x}{e^h - e^{-h}x} \sum_{\substack{i,j=1 \\ i > j}}^{N} e_{ij} \otimes e_{ji} + \frac{e^h - e^{-h}}{e^h - e^{-h}x} \sum_{\substack{i,j=1 \\ i < j}}^{N} e_{ij} \otimes e_{ji}.$$

- It satisfies the multiplicative *Yang–Baxter equation* (m-YBE),

$$R^h_{12}(x)R^h_{13}(xy)R^h_{23}(y) = R^h_{23}(y)R^h_{13}(xy)R^h_{12}(x).$$

Adapting the R-matrix: multiplicative $\rightsquigarrow$ additive

Set

$$R(u) = (R^q(x))|_{q=e^h,\, x=e^u} = (R^h(x))_{x=e^u}.$$

- The R-matrix over $\mathbb{C}[[h]]$ defined by

$$R(u) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \frac{1-e^u}{e^h - e^{u-h}} \sum_{\substack{i,j=1\\ i\neq j}}^{N} e_{ii} \otimes e_{jj} + \frac{\left(e^h - e^{-h}\right) e^u}{e^h - e^{u-h}} \sum_{\substack{i,j=1\\ i>j}}^{N} e_{ij} \otimes e_{ji} + \frac{e^h - e^{-h}}{e^h - e^{u-h}} \sum_{\substack{i,j=1\\ i<j}}^{N} e_{ij} \otimes e_{ji}.$$

- It satisfies the additive Yang–Baxter equation (a-YBE),

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

Quantized universal enveloping algebra $U(R)$

Definition

The *quantized universal enveloping algebra $U(R)$* is the h-adically completed associative algebra over $\mathbb{C}[[h]]$ generated by the elements

$$t_{ij}^{(-r)}, \qquad \text{where} \qquad 1 \leqslant i,j \leqslant N, \quad r = 1,2,\ldots,$$

subject to the defining relations

$$R_{12}(u-v)T_{13}^{+}(u)T_{23}^{+}(v) = T_{23}^{+}(v)T_{13}^{+}(u)R_{12}(u-v),$$

where

$$T_{13}^{+}(u) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes t_{ij}^{+}(u) \in \overbrace{\operatorname{End}\mathbb{C}^N}^{1} \otimes \overbrace{\operatorname{End}\mathbb{C}^N}^{2} \otimes \overbrace{U(R)}^{3}[[u]],$$

$$T_{23}^{+}(u) = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes t_{ij}^{+}(u) \in \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes U(R)[[u]],$$

$$t_{ij}^{+}(u) = \delta_{ij} - h\sum_{r=1}^{\infty} t_{ij}^{(-r)} u^{r-1}.$$

The Etingof–Kazhdan quantum affine vertex algebra

Theorem (P. Etingof, D. Kazhdan, 2000)

For any $c \in \mathbb{C}$ there exists a unique quantum vertex algebra structure over $\mathcal{V}^c(\mathfrak{gl}_N) = U(R)$ such that the vertex operator map is given by

$$Y\big(T^{+}_{[n]}(u), z\big) = T^{+}_{[n]}(u|z) T^{-}_{[n]}(u|z + hc/2)^{-1}$$

and the braiding $\mathcal{S} = \mathcal{S}(z)$ is defined by the relation

$$\mathcal{S}_{34}(z)\Big(R^{12}_{nm}(u|v|z)^{-1} T^{+24}_{[m]}(v) R^{12}_{nm}(u|v|z - hc) T^{+13}_{[n]}(u)\Big) \\ = T^{+13}_{[n]}(u) R^{12}_{nm}(u|v|z + hc)^{-1} T^{+24}_{[m]}(v) R^{12}_{nm}(u|v|z),$$

where

$$T^{\pm}_{[n]}(u) = T^{\pm}_{1\,n+1}(u_1) \dots T^{\pm}_{n\,n+1}(u_n),$$

$$T^{\pm}_{[n]}(u|z) = T^{\pm}_{1\,n+1}(z + u_1) \dots T^{\pm}_{n\,n+1}(z + u_n),$$

$$R^{12}_{nm}(u|v|z) = \prod^{\longrightarrow}_{j=1,\dots,n} \prod^{\longleftarrow}_{i=n+1,\dots,n+m} R_{ji}(z + u_j - v_{i-n}).$$

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where

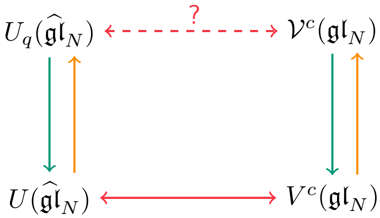
$$T^{\pm}_{[n]}(u) = T^{\pm}_{1\,n+1}(u_1) \dots T^{\pm}_{n\,n+1}(u_n),$$
$$T^{\pm}_{[n]}(u|z) = T^{\pm}_{1\,n+1}(z + u_1) \dots T^{\pm}_{n\,n+1}(z + u_n),$$
$$R^{12}_{nm}(u|v|z) = \prod_{j=1,\dots,n}^{\longrightarrow} \prod_{i=n+1,\dots,n+m}^{\longleftarrow} R_{ji}(z + u_j - v_{i-n}).$$

- P. Etingof, D. Kazhdan, 2000 $\rightsquigarrow$ rat., trig. and ellipt. R-matrix of type A
- M. Butorac, N. Jing, S. K., 2019 $\rightsquigarrow$ rat. R-matrix of types B, C and D
- S. K., 2021 $\rightsquigarrow$ trig. R-matrix of types B, C and D

$\mathcal{V}^c(\mathfrak{gl}_N)$-modules vs. $U_q(\widehat{\mathfrak{gl}}_N)$-modules?

Question

Can we establish a connection between $\mathcal{V}^c(\mathfrak{gl}_N)$-modules and $U_q(\widehat{\mathfrak{gl}}_N)$-modules?



We need 2 ingredients:

- a suitable realization of $U_q(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}[[h]] \rightsquigarrow U_h(R^h)$
- a certain deformation of $\mathcal{V}^c(\mathfrak{gl}_N)$-modules $\rightsquigarrow$ ϕ-coordinated $\mathcal{V}^c(\mathfrak{gl}_N)$-modules

The R-matrix realization of the quantum affine algebra for $\widehat{\mathfrak{gl}}_N$ (N. Reshetikhin, M. A. Semenov-Tian-Shansky, 1990)

The *quantum affine algebra $U_h(R^h)$* for $\widehat{\mathfrak{gl}}_N$ is the associative algebra generated by the central element C and elements $l_{ij}^{(r)}$ and $l_{ij}^{(-r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \ldots$, subject to the defining relations

$$R_{12}^h(x/y) L_{13}^{\pm}(x) L_{23}^{\pm}(y) = L_{23}^{\pm}(y) L_{13}^{\pm}(x) R_{12}^h(x/y),$$
$$R_{12}^h\big(xe^{hC/2}/y\big) L_{13}^{-}(y) L_{23}^{+}(x) = L_{23}^{+}(x) L_{13}^{-}(y) R_{12}^h\big(xe^{-hC/2}/y\big),$$

where $e_{ij} \in \operatorname{End} \mathbb{C}^N$ are matrix units and

$$L_{13}^{\pm}(x) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes l_{ij}^{\pm}(x) \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N \otimes U_h(R^h)[[x^{\pm 1}]],$$

$$L_{23}^{\pm}(x) = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes l_{ij}^{\pm}(x) \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N \otimes U_h(R^h)[[x^{\pm 1}]],$$

$$l_{ij}^{+}(x) = \delta_{ij} - h \sum_{r=1}^{\infty} l_{ij}^{(-r)} x^{r-1} \in U_h(R^h)[[x]],$$

$$l_{ij}^{-}(x) = \delta_{ij} + h \sum_{r=1}^{\infty} l_{ij}^{(r)} x^{-r} \in U_h(R^h)[[x^{-1}]].$$

Restricted $U_h(R^h)$-modules

Definition

An $U_h(R^h)$-module W is said to be *restricted module of level $c \in \mathbb{C}$* if

- W is topologically free as a $\mathbb{C}[[h]]$-module,
- For any $w \in W$ the expression $L^-(x)w$ has finitely many negative powers of x modulo h^n for any $n \geqslant 1$,
- The central element C acts on W as a scalar multiplication by c.

Example

The vacuum $U_h(R^h)$-module of level c is a restricted module.

Quantum current commutation relation of N. Reshetikhin, M. Semenov-Tian-Shansky, 1990

- Suppose W is a restricted $U_h(R^h)$-module of level c. The operator

$$\mathcal{L}(x)_W = L^+(x)_W L^-(xe^{hc/2})_W^{-1} \in \mathrm{Hom}(W, W((x))_h)$$

satisfies the multiplicative *quantum current commutation relation*:

$$\mathcal{L}_{13}(x)_W R_{21}^h(ye^{-hc}/x)\mathcal{L}_{23}(y)_W R_{21}^h(y/x)^{-1} \\ = R_{12}^h(x/y)^{-1}\mathcal{L}_{23}(y)_W R_{12}^h(xe^{-hc}/y)\mathcal{L}_{13}(x)_W.$$

- On the other hand, the operator

$$\mathcal{T}(u) = T^+(u)T^-(u+hc/2)^{-1} \in \mathrm{Hom}(\mathcal{V}^c(\mathfrak{gl}_N), \mathcal{V}^c(\mathfrak{gl}_N)((u))_h),$$

which defines the quantum vertex algebra structure on $\mathcal{V}^c(\mathfrak{gl}_N)$, satisfies the additive *quantum current commutation relation*:

$$\mathcal{T}_{13}(u)R_{21}(-u+v-hc)\mathcal{T}_{23}(v)R_{21}(-u+v)^{-1} \\ = R_{12}(-v+u)^{-1}\mathcal{T}_{23}(v)R_{12}(-v+u-hc)\mathcal{T}_{13}(u).$$

Modules for quantum vertex algebras

- $(\mathcal{V}, Y, \mathcal{S})$ ... a quantum vertex algebra

Definition (H.-S. Li, 2008)

A *$\mathcal{V}$-module* is a pair (W, Y_W) such that

- W is a topologically free $\mathbb{C}[[h]]$-module
- $Y_W = Y_W(\cdot, z)$ is a $\mathbb{C}[[h]]$-module map $\mathcal{V} \to \operatorname{End} W[[z^{\pm 1}]]$ which satisfies certain axioms, e.g.,

  for any $n \geqslant 0$ and $u, v \in \mathcal{V}$ there exists $r \geqslant 0$ such that for all $w \in W$

$$\left((z_1 - z_2)^r \, Y_W(u, z_1) \, Y_W(v, z_2) w \mod h^n\right)\Big|_{z_1 = z_2 + z_0}$$
$$= z_0^r \, Y_W(Y(u, z_0)v, z_2) w \mod h^n.$$

ϕ-coordinated modules for quantum vertex algebras

- $(\mathcal{V}, Y, \mathcal{S})$ ... a quantum vertex algebra
- $\phi = \phi(z_2, z_0) \in \mathbb{C}((z_2))[[z_0]]$ ... an associate of formal additive group, i.e.,

$$\phi(z_2, 0) = z_2 \quad \text{and} \quad \phi(\phi(z_2, z_0), z) = \phi(z_2, z + z_0)$$

Definition (H.-S. Li, 2011)

A *ϕ-coordinated $\mathcal{V}$-module* is a pair (W, Y_W) such that

- W is a topologically free $\mathbb{C}[[h]]$-module
- $Y_W = Y_W(\cdot, z)$ is a $\mathbb{C}[[h]]$-module map $\mathcal{V} \to \operatorname{End} W[[z^{\pm 1}]]$ which satisfies certain axioms, e.g.,

  for any $n \geqslant 0$ and $u, v \in \mathcal{V}$ there exists $r \geqslant 0$ such that for all $w \in W$

$$\left((z_1 - z_2)^r \, Y_W(u, z_1) \, Y_W(v, z_2)w \mod h^n\right)\Big|_{z_1 = \phi(z_2, z_0)}$$
$$= (\phi(z_2, z_0) - z_2)^r \, Y_W(Y(u, z_0)v, z_2)w \mod h^n.$$

Example

$\phi(z_2, z_0) = z_2 + z_0$ produces the usual notion of $\mathcal{V}$-module.

Restricted $U_h(R^h)$-modules & ϕ-coordinated $\mathcal{V}^c(\mathfrak{gl}_N)$-modules

Theorem (S. K., 2021)

Let $c \in \mathbb{C}$ *and* $\phi(z_2, z_0) = z_2 e^{z_0}$. *Then*

(irreducible) restricted $U_h(R^h)$*-modules of level* c	$=$	*(irreducible)* ϕ*-coordinated* $\mathcal{V}^c(\mathfrak{gl}_N)$*-modules.*

In particular, the corresponding ϕ*-coordinated* $\mathcal{V}^c(\mathfrak{gl}_N)$*-module map*

$$Y_W(\cdot, z)\colon \mathcal{V}^c(\mathfrak{gl}_N) \to \operatorname{End} W[[z, z^{-1}]]$$

is such that

$$Y_W(T^+(u), z) = \mathcal{L}(x)\big|_{x = ze^u}.$$

Restricted $U_h(R^h)$-modules & ϕ-coordinated $\mathcal{V}^c(\mathfrak{gl}_N)$-modules

Theorem (S. K., 2021)

Let $c \in \mathbb{C}$ *and* $\phi(z_2, z_0) = z_2 e^{z_0}$. *Then*

(irreducible) restricted $U_h(R^h)$*-modules of level* c	$=$	*(irreducible)* ϕ*-coordinated* $\mathcal{V}^c(\mathfrak{gl}_N)$*-modules.*

In particular, the corresponding ϕ*-coordinated* $\mathcal{V}^c(\mathfrak{gl}_N)$*-module map*

$$Y_W(\cdot, z)\colon \mathcal{V}^c(\mathfrak{gl}_N) \to \operatorname{End} W[[z, z^{-1}]]$$

is such that

$$Y_W(T^+(u), z) = \mathcal{L}(x)\big|_{x = ze^u}.$$

More recently, Frenkel–Jing's question was answered in full generality:

- N. Jing, F. Kong, H.-S. Li, S. Tan, 2021/22 & F. Kong, 2023
  $\rightsquigarrow$ quantum vertex algebra theory for the (un)twisted quantum affine algebras

The center of $\mathcal{V}^{-N}(\mathfrak{gl}_N)$

Definition

The *center* of the quantum vertex algebra $(\mathcal{V}, Y, \mathcal{S})$ is defined by

$$\mathfrak{z}(\mathcal{V}) = \{a \in \mathcal{V} \,:\, Y(b,z)a \in \mathcal{V}[[z]] \text{ for all } b \in \mathcal{V}\}\,.$$

- P^h ... h-permutation operator on $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$,

$$P^h = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + e^{h/2} \sum_{\substack{i,j=1\\ i>j}}^{N} e_{ij} \otimes e_{ji} + e^{-h/2} \sum_{\substack{i,j=1\\ i<j}}^{N} e_{ij} \otimes e_{ji}$$

- $P^h_{[k]} = P^h_{k-1\,k} P^h_{k-2\,k-1} \dots P^h_{1\,2}$ ... k-cycle on $\left(\operatorname{End}\mathbb{C}^N\right)^{\otimes k}$
- $D = \text{diag}\left(e^{(N-1)h/2}, e^{(N-3)h/2}, \dots, e^{-(N-1)h/2}\right)$ ... an $N \times N$ diagonal matrix

Theorem (S. K., A. Molev, 2017)

The center of $\mathcal{V}^{-N}(\mathfrak{gl}_N)$ is a commutative associative algebra topologically generated by the algebraically independent family $\Phi_m^{(r)}$ with $m = 1, \dots, N$, $r = 0, 1, \dots$ defined by

$$\Phi_m(u) = \sum_{r=0}^{\infty} \Phi_m^{(r)} u^r$$
$$= \frac{1}{h^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \operatorname{tr}_{1,\dots,k} P^h_{[k]}\, T^+_{1\,k+1}(u) \dots T^+_{k\,k+1}(u-(k-1)h)\, D_1 \dots D_k.$$

The center of $\mathcal{V}^c(\mathfrak{gl}_N)$ for $c \neq -N$

- Consider the *quantum determinant* of the matrix $T^+(u)$,

$$\mathsf{qdet}\, T^+(u) = \mathrm{tr}_{1,\dots,N}\, A^{(N)}\, T^+_{1\,N+1}(u) \dots T^+_{N\,N+1}(u-(N-1)h)\, D_1 \dots D_N,$$

where $A^{(N)}$ is the action of the normalized anti-symmetrizer

$$a^{(N)} = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \mathsf{sgn}\, \sigma \cdot \sigma \in \mathbb{C}[\mathfrak{S}_N].$$

- Its coefficients are given by

$$\mathsf{qdet}\, T^+(u) = 1 - h \sum_{r \geqslant 0} \delta_r\, u^r \in \mathcal{V}^c(\mathfrak{gl}_N)[[u]].$$

Proposition (S. K., A. Molev, 2017)

The center of $\mathcal{V}^c(\mathfrak{gl}_N)$ for $c \neq -N$ is a commutative associative algebra topologically generated by the algebraically independent family $\delta_0, \delta_1, \delta_2, \dots$.

What about the reflection equation?

- $\mathcal{S}$-locality for the vertex operator map $Y(\cdot, z)$ of $\mathcal{V}^c(\mathfrak{gl}_N)$ takes the form

$$\mathcal{T}_{13}(x)R_{21}(-x+y-hc)\mathcal{T}_{23}(y)R_{21}(-x+y)^{-1} \\ \sim R_{12}(-y+x)^{-1}\mathcal{T}_{23}(y)R_{12}(-y+x-hc)\mathcal{T}_{13}(x).$$

- $\mathcal{S}$-locality for the ϕ-coodinated module map $Y_W(\cdot, z)$ of $\mathcal{V}^c(\mathfrak{gl}_N)$ (with $\phi(z_2, z_0) = z_2 e^{z_0}$) takes the form

$$\mathcal{L}_{13}(x)R^h_{21}(ye^{-hc}/x)\mathcal{L}_{23}(y)R^h_{21}(y/x)^{-1} \\ \sim R^h_{12}(x/y)^{-1}\mathcal{L}_{23}(y)R^h_{12}(xe^{-hc}/y)\mathcal{L}_{13}(x).$$

- E. K. Sklyanin, 1988 $\rightsquigarrow$ *reflection equation* & *reflection equation algebras*
- A. Molev, E. Ragoucy, P. Sorba, 2003 $\rightsquigarrow$ *twisted q-Yangians*, a certain family of coideal subagebras in $U_q(\widehat{\mathfrak{gl}}_N)$ defined via reflection equation

$$S_{13}(x)R^q_{12}(1/xy)^{t_1}S_{23}(y)R^q_{12}(x/y)^{-1} = R^q_{12}(x/y)^{-1}S_{23}(y)R^q_{12}(1/xy)^{t_1}S_{13}(x).$$

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Question (L. Bagnoli, S. K.)

Can we interpret the reflection equation using the ϕ-coordinated module theory?

Definition

Let $\mathcal{V}$ be a topologically free $\mathbb{C}[[h]]$-module. A $\mathbb{C}[[h]]$-module map

$$\mu(z,x)\colon \mathcal{V}\otimes\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}(z,x)[[h]]$$

is said to be a *braiding map* if it satisfies

- the *quantum Yang-Baxter equation*

$$\mu_{12}(z_1,xz_2)\mu_{13}(z_1z_2,x)\mu_{23}(z_2,x)=\mu_{23}(z_2,x)\mu_{13}(z_1z_2,x)\mu_{12}(z_1,xz_2),$$

- the *unitarity condition*

$$\mu(1/z,x)\mu_{21}(z,x/z)=\mu_{21}(z,x/z)\mu(1/z,x)=1,$$

- the *singularity constraint*

$$\mu_\pm(z_1/z_2,z_2)(u\otimes v)\in V\otimes V\otimes\mathbb{C}[z_1,z_1^{-1}]((z_2^{\pm 1}))[[h]] \text{ for all } u,v\in\mathcal{V},$$

  where

$$\mu_\pm(z_1/z_2,z_2)(u\otimes v)=\big(\iota_{z,x^{\pm1}}\,\mu(z,x)(u\otimes v)\big)\,\big|_{z=z_1/z_2,\,x=z_2}.$$

Definition

Let $\mathcal{V}$ be a topologically free $\mathbb{C}[[h]]$-module. A braiding map

$$\mu(z,x)\colon \mathcal{V}\otimes\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}(z,x)[[h]]$$

is said to be *compatible* with respect to a $\mathbb{C}[[h]]$-module map

$$\nu(z,x)\colon \mathcal{V}\otimes\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}(z,x)[[h]]$$

if

- the map ν is invertible, i.e. there exists a map

  $$\nu^{-1}(z,x)\colon \mathcal{V}\otimes\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}(z,x)[[h]]$$

  satisfying

  $$\nu(z,x)\nu^{-1}(z,x)=\nu^{-1}(z,x)\nu(z,x)=1,$$

- the map

  $$\sigma(z,x)=\nu(z,x)\mu(z,x)\nu_{21}^{-1}(1/z,zx)$$

  is again a braiding map.

The pair (μ,ν) is then said to be a *compatible pair*.

Definition

Let $(\mathcal{V}, Y, \mathcal{S})$ be a quantum vertex algebra, (μ, ν) a compatible pair on $\mathcal{V}$ such that

$$\widehat{\mathcal{S}}(z) = \nu(z,x)\mu(z,x)\nu_{21}^{-1}(1/z, zx)$$

and W a topologically free $\mathbb{C}[[h]]$-module equipped with a $\mathbb{C}[[h]]$-module map

$$\begin{aligned} Y_W(\cdot, z)\colon \mathcal{V} \otimes W &\to W[[z, z^{-1}]], \\ v \otimes w &\mapsto Y_W(v,z)w = \sum_{r\in\mathbb{Z}} v_r w z^{-r-1}. \end{aligned}$$

A pair (W, Y_W) is said to be a *(μ,ν)-deformed ϕ-coordinated $\mathcal{V}$-module* (with $\phi(z_2, z_0) = z_2 e^{z_0}$) if the map $Y_W(\cdot, z)$ satisfies

- the *weak ν-associativity property*: for any elements $u, v \in \mathcal{V}$ and $n \in \mathbb{Z}_{>0}$ there exists a nonzero polynomial $p(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ such that we have

$$\begin{aligned} &\left(p(z_1,z_2)Y_W(u,z_1)Y_W(v,z_2) \mod h^n\right)\big|_{z_1=z_2e^{z_0}} \\ &- p(z_2e^{z_0}, z_2)Y_W\left(Y(z_0)\nu(z_0,z_2)(u\otimes v), z_2\right) \in h^n \operatorname{Hom}(W, W[[z_0^{\pm1}, z_2^{\pm1}]]), \end{aligned}$$

- the *μ-locality property*: for any $u, v \in \mathcal{V}$ and $n \in \mathbb{Z}_{>0}$ there exists a nonzero polynomial $q(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ such that we have

$$\begin{aligned} &\Big(q(z_1,z_2)Y_W(z_1)\big(1\otimes Y_W(z_2)\big)\big(\mu(z_1/z_2, z_2)(u\otimes v)\otimes w\big) \\ &\quad - q(z_1,z_2)Y_W(v,z_2)Y_W(u,z_1)w\Big) \in h^n W[[z_1^{\pm1}, z_2^{\pm1}]] \quad \text{for all } w \in W, \end{aligned}$$

- $\dots$.

Orthogonal twisted h-Yangian $\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)$

Definition (A. Molev, E. Ragoucy, P. Sorba, 2003)

The *orthogonal twisted h-Yangian* $\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)$ is the h-adically completed algebra generated by the elements $s_{ij}^{(r)}$, where $i,j = 1,\dots,N$ and $r = 0,1,\dots,$ subject to the defining relations

$$R_{12}^h(x/y)S_{13}(x)R_{12}^h(1/xy)^{t_1}S_{23}(y) = S_{23}(y)R_{12}^h(1/xy)^{t_1}S_{13}(x)R_{12}^h(x/y),$$
$$s_{ij}^{(0)} = 0 \text{ for } 1 \leqslant i < j \leqslant N \qquad \text{and} \qquad s_{ii}^{(0)} = 1 \text{ for } i = 1,\dots,N.$$

The matrix $S(u)$ is defined by

$$S(u) = \sum_{i,j=1}^{N} e_{ij} \otimes s_{ij}(u),$$

where for all $i,j = 1,\dots,N$ its matrix entries $s_{ij}(u)$ are given by

$$s_{ii}(u) = 1 + h\sum_{r\geqslant 1} s_{ii}^{(r)}u^{-r} \qquad \text{and} \qquad s_{ij}(u) = h\sum_{r\geqslant 0} s_{ij}^{(r)}u^{-r} \quad \text{if} \quad i \neq j.$$

$\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)$ as a (μ,ν)-deformed ϕ-coordinated $\mathcal{V}^c(\mathfrak{gl}_N)$-module

Theorem (L. Bagnoli, S. K., 2024)

For any $c \in \mathbb{C}$ there exists a compatible pair (μ,ν) over $\mathcal{V}^c(\mathfrak{gl}_N)$ such that we have

- *There exists a unique structure of (μ,ν)-deformed ϕ-coordinated $\mathcal{V}^c(\mathfrak{gl}_N)$-module on $\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)$ such that the module map is given by*

$$Y_{\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)}(T^+_{[n]}(u_1,\ldots,u_n),z) = S_{[n]}(ze^{u_1},\ldots,ze^{u_n}),$$

where $S_{[n]}(ze^{u_1},\ldots,ze^{u_n})$ stands for the ordered product

$$\prod_{i=1,\ldots,n}^{\longrightarrow} S_i(ze^{u_i}) R_{i\,i+1}(1/z^2 e^{u_i+u_{i+1}})^{t_i} \ldots R_{in}(1/z^2 e^{u_i+u_n})^{t_i}.$$

- *The coefficients of the (renormalized) Sklyanin determinant*

$$Y_{\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)}(\operatorname{qdet} T^+(0),z) = \operatorname{tr}_{1,\ldots,N} A^{(N)} S_{[N]}(z, ze^{-h},\ldots,ze^{-(N-1)h}) D_1 \ldots D_N$$

belong to the center of the algebra $\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)$.

Remark

The second statement was originally proved by A. Molev, E. Ragoucy, P. Sorba in 2003.

Theorem (L. Bagnoli, S. K., 2024)

For any $c \in \mathbb{C}$ there exists a compatible pair (μ, ν) over $\mathcal{V}^c(\mathfrak{gl}_N)$ such that we have

- *Let W be a topologically free $\mathbb{C}[[h]]$-module and also a $\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)$-module. There exists a unique structure of (μ, ν)-deformed ϕ-coordinated $\mathcal{V}^c(\mathfrak{gl}_N)$-module on W such that the module map satisfies*

$$Y_W(T_{[n]}^+(u_1, \ldots, u_n), z) = S_{[n]}(ze^{u_1}, \ldots, ze^{u_n})_W.$$

- *Let W be a (μ, ν)-deformed ϕ-coordinated $\mathcal{V}^c(\mathfrak{gl}_N)$-module such that*

$$Y_W(t_{ij}^{(-1)}, z) \in \mathrm{Hom}(W, z^{-1}W[[z^{-1}]]) \quad \textit{for all } 1 \leqslant i \leqslant j \leqslant N.$$

  There exists a unique structure of $\mathrm{Y}_h^{\mathrm{tw}}(\mathfrak{o}_N)$-module on W such that

$$S(z)_W = Y_W(T^+(0), z).$$

Thank you