

Cohomology of symplectic spinor twisted de Rham sequence and its subcomplexes

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Motivation and Content

Analysis of **vector-valued functions** on manifolds (i.e. for deRham operator, signature operator, Dirac operators) or of sections of vector bundles \implies

Analysis of functions with values in infinite dimensional vector (Hilbert, Fréchet) spaces (i.e. symplectic Dirac operator, de Rham complex twisted by "**symplectic spinors**", **introduced, and possible subcomplexes**)

- A) Symplectic spinors
- B) Geometric Quantization
- C) Symplectic Dirac operator and its associated operator
- D) Symplectic twistor operators and their complexes

A) Symplectic Spinors - Bargmann–Fock picture

1) Vector spaces

- (V, ω_0) symplectic vector space over \mathbb{R} of dimension $2n$ (phase space)
- J_0 compatible (= symplectic) complex structure (i.e., $J_0^2 = -\text{Id}_V$), inducing positive definite bilinear form g_0 and norm $||_0$ on V induced by g_0
- $H_{BF} = \{f : V \rightarrow \mathbb{C} \text{ holomorphic} \mid \int_V f(v) e^{-|v|_0^2} d\lambda_{V, g_0} < \infty\}$
- **Bargmann–Fock space**, holomorphic:
 $j_{0,2} \circ (Df)(x) = (Df)(x) \circ J_0$ for each $x \in V$, where $j_{0,2}$ standard complex structure on $\mathbb{R}^2 \simeq \mathbb{C}$

A) Symplectic Spinors - Bargmann–Fock picture

2) Groups, representations, and further vector spaces

- $G = Sp(V, \omega_0)$ admits a connected $2 : 1$ covering with covering map $\lambda : \tilde{G} = Mp(V, \omega_0) \rightarrow G$ (non-matrix reductive Lie group)
- **Segal–Shale–Weil/oscillator/metaplectic representation:**
 $\rho : \tilde{G} \rightarrow \mathcal{U}(H_{BF})$ injective homomorphism, which is, moreover, a topological group representation, i.e., the induced map $\tilde{G} \times H_{BF} \rightarrow H_{BF}$ is continuous, $\text{Im } \rho \simeq Mp(V, \omega_0)$,
 $H_{BF} = H_{BF}^+ \oplus H_{BF}^-$ [Weil], [Shale], see also [Robinson, Rawnsley] and [Habermann, Habermann]
- $S = \bigcap_{k=0}^{\infty} \{f \in H_{BF} \mid \int_V f(v)(1 + |v|_0^2)^{k/2} e^{-|v|_0^2} d\lambda_{V, g_0} < \infty\}$ - smooth vectors of ρ , $S = S^+ \oplus S^-$
- $\mathbb{S} = \bigoplus_{k \geq 0} \odot^k \mathbb{W}$ (finitely supported sum) - K -finite smooth vectors of ρ , **Fock space**, where $\mathbb{W} = \text{Ker}(J_0^c - i\text{Id}_{V^c})$ and $J_0^c : V^c = V \otimes \mathbb{C} \rightarrow V^c$ is the complexification of J_0

A) Symplectic spinors - Lagrangian picture

Let L be a Lagrangian subspace of (V, ω_0) i.e., maximal and ω_0 -isotropic.

If $\lambda(\tilde{J}_0) = J_0 \in \text{Aut}(V)$, then $\rho(\tilde{J}_0)f = \pm \mathcal{F}f$, where $(\mathcal{F}f)(y) = \int_{x \in L} e^{-2\pi i g_0(x,y)} f(x) d\lambda_{L, g_0}(x)$, where $y \in L$, is the Fourier transform and $f \in L^2(L)$.

Moreover, the Lie algebra representation satisfies

$$(\rho_*)_e(\tilde{J}_0)f = k_1 \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} - k_2 \sum_{i=1}^n x_i^2 f,$$

where $k_i \in \mathbb{C}$, $i = 1, 2$, i.e., it acts as the quantum Hamiltonian of a harmonic oscillator.

B) Metaplectic structure - analogue of spin structure

- 1) (M, ω) symplectic manifold of dimension $2n$, \mathcal{P} principal \tilde{G} -bundle with a bundle map $\Lambda : \mathcal{P} \rightarrow \mathcal{R} = \{b \mid b \text{ symplectic basis of } (T_m M, \omega_m), m \in M\}$ which is "almost" a principal bundle map:
 $\Lambda(b' \cdot g) = \Lambda(b') \cdot \lambda(g)$ for all $b' \in \mathcal{P}$ and $g \in \tilde{G}$. The pair (\mathcal{P}, Λ) is called a **metaplectic structure** (B. Kostant).
Existence and number of isomorphism classes well known: 2nd Stiefel–Whitney $w_2(TM)$ vanishes or 1st Chern class $c_1(T^c M)$ is even (existence); $H^1(M, \mathbb{Z}_2)$ (number of classes) [Forger, Hess].
- 2) $SM = \mathcal{P} \times_\rho S \rightarrow M$, locally trivial smooth bundle of Fréchet vector spaces, $\Gamma(SM)$ smooth sections of $SM \rightarrow M$ - **Schwartz symplectic spinor fields**.
- 3) $\Gamma(\mathbf{SM}) := C^\infty(\mathcal{P}, H_{BF})^{\tilde{G}}$ **Bargmann–Fock symplectic spinor fields**. The bundle $\mathcal{H}_{BF} := \mathcal{P} \times_\rho H_{BF}$ need not be smooth [KryslCMP]. So we "cannot talk" about smooth sections.

B) Geometric Quantization via Symplectic Spinors Fields

- a) $f \in C^\infty(M)$, df 1-form, $X_f = \sharp_\omega(df)$ Hamiltonian vector field of f , $\{f, g\} := \omega(X_f, X_g)$ Poisson bracket, $\omega(\sharp_\omega \alpha, Y) = \alpha(Y)$
- b) $q : C^\infty(M) \rightarrow \text{End}(\Gamma(SM))$
 $q(f)(\phi) = -i\hbar \mathcal{L}_{X_f} \phi$, $f \in C^\infty(M)$, $\phi \in \Gamma(SM)$

Theorem ([Habermann, Klein]): $q(\{f, g\}) = i\hbar[q(f), q(g)]$, where $[,]$ is the commutator of endomorphisms (Heisenberg commutation relation).

Thus, q is a homomorphism of Lie algebras, up to a constant.

B) Lie derivative of symplectic spinor fields

X vector field $\implies \theta_t, t \in (-\epsilon, \epsilon)$ - local flow of X (1-parameter group of diffeomorphisms) \implies

$\tilde{\theta}_t(e_1, \dots, e_{2n}) = ((\theta_t)_*e_1, \dots, (\theta_t)_*e_{2n}); \tilde{\theta}_t : \mathcal{R} \rightarrow \mathcal{R},$
 $(e_1, \dots, e_{2n}) \subseteq T_m M$ symplectic basis in point m

$\bar{\theta}_t : \mathcal{P} \rightarrow \mathcal{P}$ a lift of $\tilde{\theta}_t$ to \mathcal{P} (the metaplectic structure)

$\mathcal{L}_X \phi = \frac{d}{dt} \Big|_{t=0} ((\bar{\theta}_t^{-1})_* \phi)$, where

ϕ is understood as $\phi \in C^\infty(\mathcal{P}, H_{BF})^{\tilde{G}}$.

B) More familiar - with analytic difficulties

Hilbertized canonical case: $M = \mathbb{R}^{2n}$, Fréchet space S replaced by Hilbert space H_{BF} , product bundle $\mathbb{R}^{2n} \times H_{BF} \rightarrow \mathbb{R}^{2n}$

Instead of $\Gamma(SM) = C^\infty(\mathbb{R}^{2n}, S)$, "hilbertize" again: $L^2(\mathbb{R}^{2n}, H_{BF})$ (square integrable H_{BF} -valued functions) $\simeq L^2(\mathbb{R}^{2n}) \hat{\otimes}_{\mathbb{R}} H_{BF} \simeq H_{BF}$ since H_{BF} is separable (countable Hilbert 'basis')

Thus $q : C^\infty(M) \rightarrow \text{End}(H_{BF})$ as we are used to.

Why doesn't it contradict the Groenewold–Van Hove theorem? As described, e.g., in [Guillemin, Sternberg]; or [Gotay - arxiv 1998]

C) Symplectic connections and associated derivatives

- ∇ **symplectic connection** on (M, ω) , i.e., $\nabla\omega = 0$ with possibly non-zero torsion. Space of symplectic and torsion-free connections is modeled on the infinite dim. vector space $\Gamma(S^3 T^*M)$ [Libermann], [Tondeur], [Gelfand, Retakh, Shubin]. ∇ lifts to a principal connection on \mathcal{R} and on \mathcal{P} and defines the associated connection ∇^S on the induced vector bundle $SM \rightarrow M$, $\nabla^S : \Gamma(SM) \rightarrow \Gamma(SM \otimes T^*M)$.
- Let J be a compatible almost complex structure on M inducing a Riemannian metric g on M
- $SM \otimes T^*M \simeq SM \otimes TM$ isomorphic by
 $i_1 = \text{Id}_{SM} \otimes \sharp_\omega$ or by
 $i_2 = \text{Id}_{SM} \otimes \sharp_g$

C) Symplectic Dirac operators

It is known (e.g., [Kadlcakova], [KryslJOLT])

$H_{BF} \otimes V \simeq H_{BF} \oplus T_{BF}$, where T_{BF} is a sum of two irreducible representations of \tilde{G} . Let $p : V \otimes H_{BF} \rightarrow H_{BF}$ be the projection onto H_{BF} along T_{BF}

Definition:

$D = p \circ i_1 \circ \nabla^S$ symplectic Dirac operator

$\tilde{D} = p \circ i_2 \circ \nabla^S$ second symplectic Dirac

Associated operator: $P = i[\tilde{D}, D]$

P is of second order. Torsion of $\nabla = 0 \implies$ it is self-adjoint
[Habermann, Habermann] with a principal symbol $g(\xi, \xi)\text{Id}_{SM}$

C) Metaplectic structures on the 2-sphere

Associated operator P on the sphere:

Round S^2 with volume form (or $\simeq CP^1$ with Fubini–Study form, symplectomorphic). First Chern number is 2 (even), thus S^2 admits a metaplectic structure.

Number of inequivalent metaplectic structures is $|H^1(S^2, \mathbb{Z}_2)| = |\{0\}| = 1$. Thus the metaplectic structure is trivial, isomorphic to $S^2 \times Mp(\mathbb{R}^2, \omega_0)$.

C) Spectrum of associated operator on the 2-sphere

Spectrum of P on S^2 ([Habermann, Habermann]):

$\{4(l+j+1)^2 - 3(2l+1)^2 - 1 \mid (l,j) \in \mathbb{N}_0^2\}$ (with multiple occurrence and possible non-trivial multiplicity)

Connected to the **Pell's equation** (cattle problem)

The kernel of P is infinite dimensional as is its co-kernel (target quotiented by image), thus P is not Fredholm

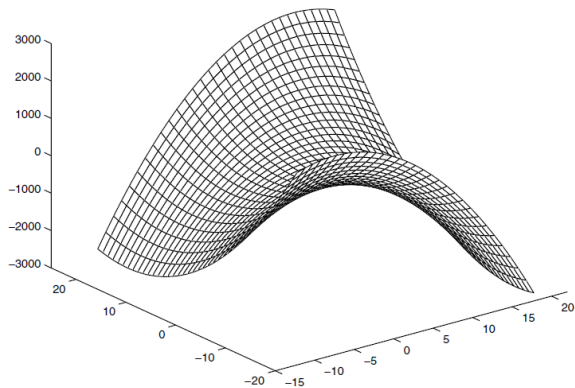
$P = \bigoplus_{l \geq 0} P_l$, where P_l is a differential operator on sections of complex line bundles on S^2

$\text{Ker } P_l$ is non-trivial iff there exists integers i, j such that (l, j) is (l_i, j_i) where $(l_0, j_0) = (0, 0)$,

$(l_i, j_i) = (11l_{i-1} + 4j_{i-1} + 7, 8l_{i-1} + 3j_{i-1} + 5)$ for $i \in \mathbb{N}$. Eg., we have $(l_1, j_1) = (7, 5)$ $\dim \text{Ker } P_7 = 13$, and $(l_2, j_2) = (104, 76)$

$\dim \text{Ker } P_{104} = 181$

C) Spectrum of P on the 2-sphere



D) Tensor product decomposition into irreducibles

Let \mathfrak{g} be the Lie algebra of the metaplectic group \widetilde{G} , \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and let us choose a system of positive roots of $(\mathfrak{g}, \mathfrak{h})$. Then the system of fundamental weights $\{\varpi_i\}_{i=1}^n$ is determined.

We denote the irreducible module with the highest weight

$$\lambda = \sum_{i=1}^n \lambda_i \varpi_i \text{ by } L(\lambda_1, \dots, \lambda_n).$$

Let us set $k_{n,i} = n - |n - i|$,

$\Xi = \{(i, j) \mid i = 0, \dots, n, j = 0, \dots, k_{n,i}\}$ (Ξ has a triangular shape),

$$\text{sgn}(+) = 0, \text{sgn}(-) = 1.$$

D) Highest weights in Symplectic spinor twisted deRham

Space of smooth K -finite vectors, $K \simeq \widetilde{U(n)}$, of the space of symplectic spinors H_{BF}^\pm is isomorphic to $L(\underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_j) \oplus L(-\frac{1}{2}, \dots, -\frac{1}{2}, -\frac{3}{2})$ as a \mathfrak{g} -module, \mathfrak{g} Lie algebra of \widetilde{G} or of G .

$E_{ij}^\pm = L(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_j, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{n-j-1}, -1 + \frac{1}{2}(-1)^{i+j+\text{sgn}(\pm)})$ for

$i = 0, \dots, n-1, j = 0, \dots, k_{n,i}$ and $i = n, j = 0, \dots, n-1$. For $i = j = n$, we set $E_{nn}^+ = L(\frac{1}{2}, \dots, \frac{1}{2})$ and $E_{nn}^- = L(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{5}{2})$. For $i = n+1, \dots, 2n$ and $j = 0, \dots, k_{n,i}$, we set $E_{ij}^\pm = E_{(2n-i)j}^\pm$.

Theorem: The following decomposition into irreducible \mathfrak{g} -modules holds

$$\bigwedge^i V \otimes S^\pm = \bigoplus_{(i,j) \in \Xi} E_{ij}^\pm.$$

Proof. [KryslJOLT2] based on Britten, Hooper, Lemire [BHL]; character formulas (Jantzen).

Symplectic Twistor operators

(M, ω) , ∇ symplectic connection, ∇^S associated connection

Further associated operators are **symplectic twistor operators**:

$$T_{00} = (\text{Id}_{\Gamma(SM \otimes T^*M)} - \rho) \circ \nabla^S$$

$T_{ij} = p_{i+1, j+1} \nabla_{\Gamma(E^{ij})}^S$, where $E^{ij} \subseteq \wedge^i T^*M \otimes SM$ is induced by

the sums of two irreducible representation of \tilde{G} ; p_{ij} is the unique orthogonal projection onto it; j parametrizes irreducible representations in $\wedge^i V \otimes S$, $j = 0, \dots, k_{n,i}$ [KryslJOLT].

Theorem [KryslAACA]: For each (i, j) , $(T_{i+k, j+k})_{k \in \mathbb{Z}}$ is a complex if ∇ is torsion-free and its symplectic Weyl curvature vanishes.

For which manifolds with symplectic connections such complexes exist, i.e., which manifolds admit connections which are symplectically Weyl-flat?

D) Symplectic Weyl-flat connections

- $2(n+1)\sigma_{ijkl} := \omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega_{jk}\sigma_{il} + 2\sigma_{ij}\omega_{kl}$
- " $W := R - \sigma$ "- **symplectic Weyl tensor** ($W = 0$ - symplectic Weyl-flat/Ricci type) [Vaisman], Cahen, Gutt, Schwachhoefer, Bieliavsky
- Suppose that (M, ω) admits a **compatible Kähler structure** J and ∇ is the Levi-Civita connection of the induced g . ∇ is symplectic Weyl-flat iff its so-called holomorphic sectional curvature is constant. If in addition, the manifold is geodesically complete, it is covered by the complex projective space, by the complex plane or by the open unit ball. The covering is a holomorphic isometry. See [Kobayashi–Nomizu] or [Lee].
- specific bipolarized, bi-Lagrangian manifolds are Weyl-flat [Vaisman]; Kodaira–Thurston mnfld [Milnor], [Fox]

D) Metaplectic structures on symplectic Weyl-flat mnfd.

- If covered by Kähler: $w_2(TX)$ for $X = \mathbb{C}P^{2k+1}$, \mathbb{C}^n and B^{2n} is null and thus, these symplectic manifolds admit a metaplectic structure. Since $w_2(T\mathbb{C}P^{2k}) \neq 0$, $\mathbb{C}P^{2k}$ does not admit. If $p : X \rightarrow M$ is a covering, $p^*w_2(TM) = w_2(TX) \implies M$ does not admit a metaplectic structure if covered by $\mathbb{C}P^{2k+1}$ and it admits in any other cases since p^* is injective.
- Since the Kodaira–Thurston manifold is symplectomorphic to the quotient of a Lie group (Heisenberg group) by a discrete subgroup, it is parallelizable. It admits a metaplectic structure.

D) Ellipticity and Further properties

Definition: A complex is elliptic if its principal symbol sequence is exact out of the zero section of the cotangent bundle.

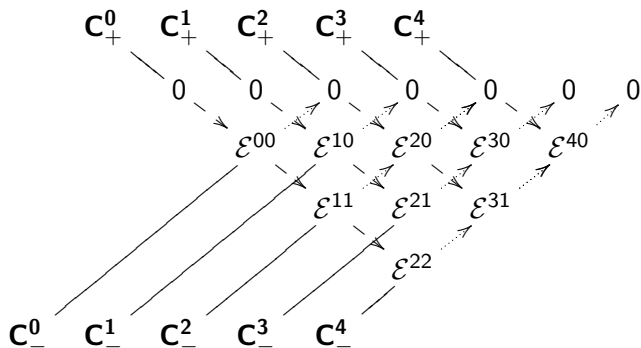
Theorem: [KryslJGP26]: Let (M, ω, ∇) be a symplectic Weyl-flat manifold. Then the truncated complexes of symplectic twistor operators are elliptic.

Truncation means: stop the complex at position $(i + \lfloor \frac{2n-i}{2} \rfloor, \lfloor \frac{2n-i}{2} \rfloor)$

Theorem: If the symplectic Weyl curvature of ∇ vanishes, $\omega^{\wedge r} \wedge$ is a well defined map between the cohomology groups $H^i(T_{i+k, j+k}) \rightarrow H^{i+2r+k}(T_{i+2r+k, j+k})$ of the complexes.

Complexes of twistor operators - elliptic parts

$\mathcal{E}^{ij} := \mathcal{P} \times_{\tilde{G}} E^{ij}$, associated bundles to the metaplectic bundle \mathcal{P}



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