SINGULAR FOLIATION

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An invitation + Neighborhoods of leaves

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I] What ? How to define ?

II] Why ? Where do they appear ?

III] Neighborhood of leaves.

Is there a lot of them ?

I] What are they ?

What is a singular foliation ? A first attempt

A first attempt to define singular foliations on M :

Definition

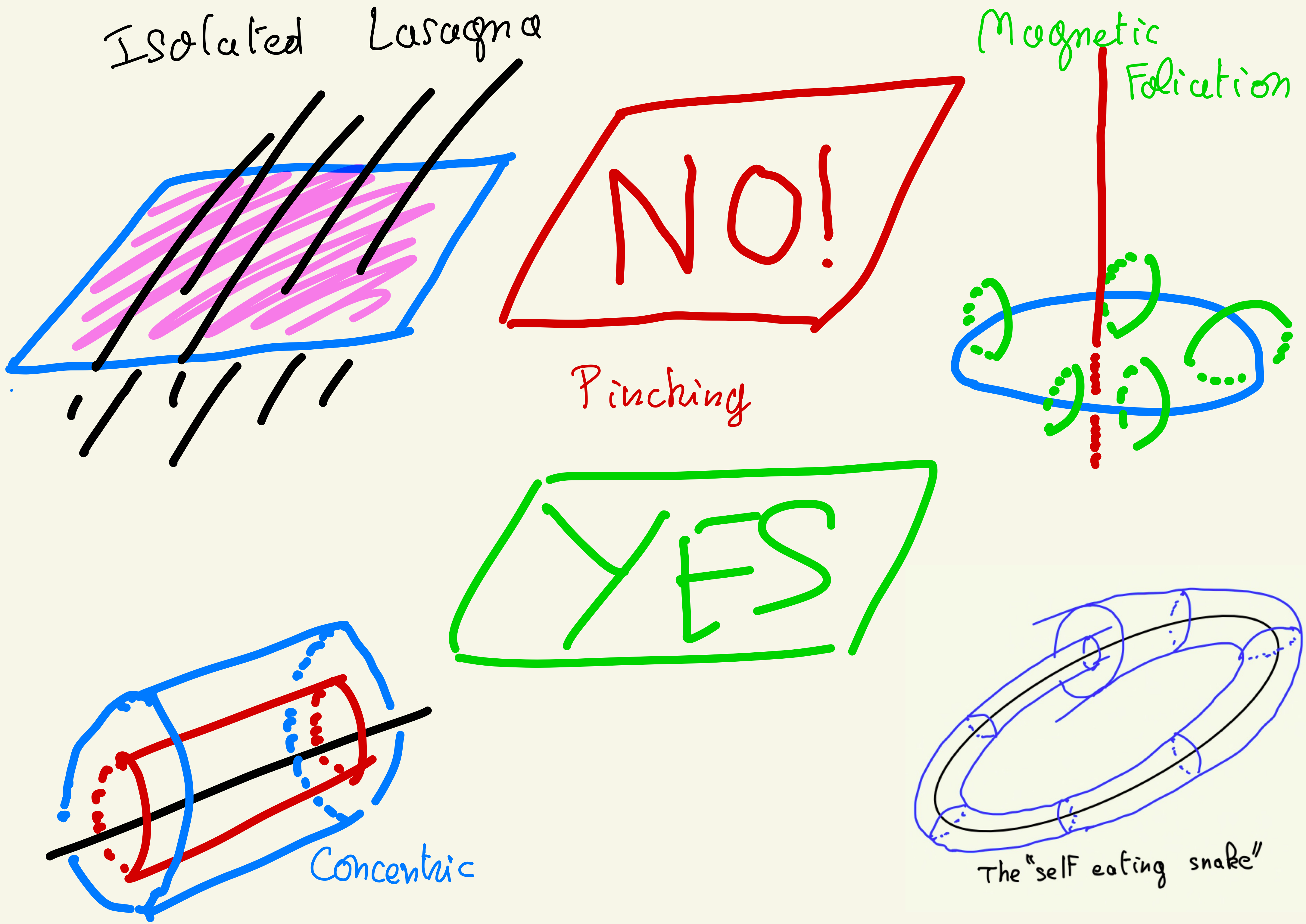
A *partitionifold* of M is a partition of M into connected immersed submanifolds[a], called leaves.

a. From now on, "submanifold" means by default "immersed submanifolds".

Notation $L_\bullet : m \mapsto L_m$.

Question

Should we take it as a definition of singular foliation ?


Isolated Lasagna
NO!
Pinching
Magnetic Foliation
YES
Concentric
The "self eating snake"

Dynamic

Much better: smooth partitionifold.

Definition

A partitionifold $L_\bullet$ is said to be <u>smooth</u> if for every $\ell \in M$ and every tangent vector $u \in T_\ell L_\ell$, there exists a vector field X through u which is tangent to all leaves.

This forbids isolated lasagnas, magnetic or weird partitionifolds. It is better.

Question

Should we take it as a definition of singular foliation ?

Why not ?

The flow of a vector field tangent to all leaves preserves $L_\bullet$.

Proposition

Let $L_\bullet$ be a smooth partitionifold.

1. *Travelling along a leaf is boring*
2. *Every leaf has a transverse structure*
3. *Which is unique*
4. *And there is a Weinstein-splitting theorem.*

More explicit maybe ?

The flow of a vector field tangent to all leaves preserves $L_\bullet$.

Proposition

Let $L_\bullet$ be a smooth partitionifold.

1. *Two points on the same leaves have open neighborhoods on which $L_\bullet$ are isomorphic.*
2. *For Σ transverse to L, $m \mapsto \Sigma \cap L_m$ is a smooth partitionifold.on a neighborhood of $L \cap \Sigma$.*
3. *And any two such transverse smooth partitionifolds have isomorphic germs.*
4. *And near any point m, $L_\bullet$ is a isomorphic to the direct product of the leaf by any representative of the transverse structure*

So is it good?

YES, Chow theorem.

NO, It has lost (so far)

Definition

A singular foliation on a smooth manifold M is a subspace $\mathcal{F} \subset \mathfrak{X}_c(M)$ which

(α) is involutive,

(β) is a $\mathcal{C}^\infty(M)$-module

(γ) is locally finitely generated.

← This definition has won!

Wait, this has no leaves!

Let $(M, \mathcal{F})$ be as above

Def 1

Leaves are equivalence classes of $\sim$ given by

$m_0 \sim m_1$ iff $X_1 \dots X_k \in \mathcal{F}$

s.t.

$$\varphi_1^{X_1} \circ \dots \circ \varphi_1^{X_k}(m_0) = m_1$$

Def 2

~~Def:~~ $T_m \mathcal{F} = \{ X(m) \mid X \in \mathcal{F} \}$

Leaves are submanifolds $L \subseteq M$

s.t. $T_m L = T_m \mathcal{F}$

for all $m \in M$

(+ maximal)

Theorem: (Hermann (55), Nagoya (75), Androulidakis - Skandalis (2003))

Let $(M, \mathcal{F})$ be a singular foliation.

1) Def 1 $\Longleftrightarrow$ Def 2

2) and leaves form a smooth partitionifold of M.

Singular Foliations $\longrightarrow$ Smooth partitionifolds

NOT INJECTIVE

NOR SURJECTIVE.

What are the examples?

Examples

Suspension

{Poisson manifolds}

Image of anchor map of a Lie algebroid

{Lie group actions}

{Coisotropic submanifolds}

Differential Operators

Vector fields vanishing at a given order at a given point

{Vector fields tangent to $W \subseteq M$}

Around (Affine) Sub-Varieties W

{Vector fields vanishing on W}

{Vector fields tangent to fibers $M \to N$}

A pure scandal!

<u>Open question:</u> Are vector fields on $\mathbb{R}^2$ vanishing at order 2 at 0 the image through the anchor map of a Lie algebroid. ?

<u>Open question:</u> Is any S.F. the image of a Lie algebroid (near any point) ?

<u>A</u> Through $\mathbb{Q}$-manifolds! $\begin{pmatrix} \text{Laurent} \\ \text{-Strobl} \\ \text{-CLG} \end{pmatrix}$

An overview.

Androulidakis – Skandalis

Debord showed leafwise smoothness.

holonomy groupoid

its C^* algebra

\+ showed

* Index theorems...

* Pseudo ellipticity...

* K-K theory...

* YUNCKEN

* MOHSEN

∞-groupoids LOUIS

Classifi-cation

Deformations (Lavau)

Characteristic classes

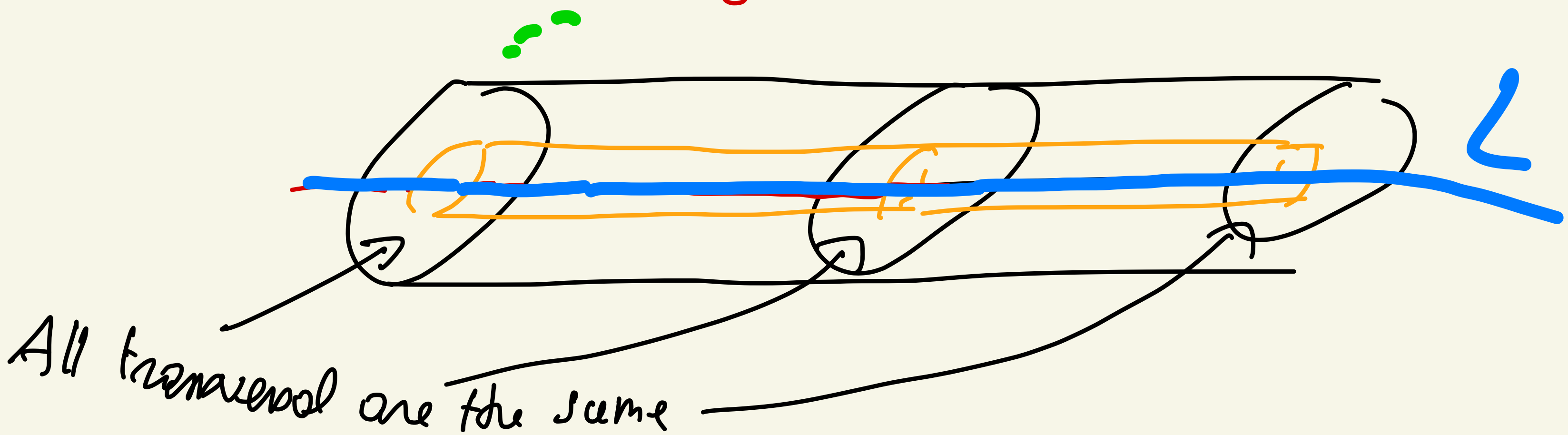
II Neighborhoods of leaves

(Low Tech.)

$(M, \mathcal{F})$ singular Foliation

1) Leaves make sense

2) Transverse singular Foliations make sense



About transversals

Question: Knowing L a leaf of S.

$(\mathbb{D}^k, \mathcal{F}_0')$ a transverse model

how many singular foliation can we have?

* For which $\{0\}$ has to be a leaf.

A

- For regular Foliation

  * As many as maps

  $$\pi_1(L) \longrightarrow \mathrm{Diff}_0(\mathbb{R}^k)$$

- If leaf is S^1, all suspension

  Needed: Outer Symmetry of $(\mathbb{D}_0, \mathcal{F}_0)$

Scandal again!

- $L = S^2$ $\qquad L = \mathbb{T}^2$

Nobody knew!

Theorem: (CLG, Roubtsov, Kuchin) If $\pi_1(L) = 0$ and $\mathcal{Z}_0$ is made of vector field with no linear part, then, at formal level, there is only ONE (the trivial product).

(+ some linearization results.
Beauville-Levi-Malcev's theorem.)

Some vocabulary

Let $(M, \mathcal{F})$ be a singular Foliation:

DeF: 1) Inner-Symmetry (Inn)

(at Formal level) 2) Symmetry.

3) Outer symmetry (Out)

$$\mathrm{Inn}(\mathfrak{h}_0) \longrightarrow \mathrm{Sym}(\mathfrak{h}_0) \xrightarrow{?} \mathrm{Out}(\mathfrak{h}_0)$$

Example: Let $(M, \mathcal{F})$ be a singular Foliation choose a leaf L

The formal setting

Thm: (CLG, Simon Fischer) Given $L + (D^n, \tau_0)$

$\exists$ 1-1 correspondence between

(i) ∞-jets of S.F. with leaf L and transverse τ

(ii) Triplets made of

- A Galois cover $\tilde{L} \to L$
- + An extension $\text{Inner} \to H \to \pi_1(\tilde{L}, L)$ with $H \subseteq \text{Sym}(\tau)$
- + An H-principal bundle

Let Inner_2 be flows of vector fields in $\mathbb{C}$ with no linear part.

$$\mathrm{Inner}_2 \longrightarrow H \longrightarrow \frac{H}{\mathrm{Inner}_2}$$

has contractible fiber

Coro: H-ppal bundle can be replaced by $\dfrac{H}{\mathrm{Inner}_2}$-ppal bundle

Finite dimensional Lie group.

Coro: If $\pi_1(L)=0$, $\exists$ 1-1 correspondence

(i) ∞-jets of S.F. with leaf L and transversal $\mathbb{C}$

(ii) $\dfrac{\mathrm{Inner}}{\mathrm{Inner}_{\geq 2}}$ - ppal bundles over L

Corollary 3.12. *Let L be a manifold and $(\mathbb{R}^d, \mathscr{T}_0)$ be a formal singular foliation near 0 which vanishes quadratically. There is a one-to-one correspondence between the three following sets:*

(i) *singular foliations along a leaf L with a transverse model $(\mathbb{R}^d, \mathscr{T}_0)$, and*

(ii) *pairs made of a Galois cover $\tilde{L} \to L$ together with a group extension of the form*

$$\mathrm{Inner}(\mathscr{T}_0) \hookrightarrow H \to \pi_1(\tilde{L}, L)$$

where $H \subset \mathrm{Sym}(\mathscr{T}_0)$, up to conjugation by an element in $\mathrm{Sym}(\mathscr{T}_0)$.

(iii) *group morphisms $\pi_1(L) \to \mathrm{Out}(\mathscr{T}_0)$, up to conjugation by an element of $\mathrm{Out}(\mathscr{T}_0)$.*

<u>**Coro**</u> SF over $\mathbb{T}^2$ are given by

- Two outer symmetries $\overline{\varphi}, \overline{\psi}$
- An element of universal cover of Inn

$$\varphi \psi \varphi^{-1} \psi^{-1} = k(1)$$

where $\varphi, \psi, k(t) : [0,1] \longrightarrow \mathbb{R}$ are representatives

Děkuji vám