Minimal unital cyclic C_∞-algebras and the real and rational homotopy types of closed manifolds

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OUTLINE

1. Models of RHT and real homotopy theory.

2. Hodge homotopy and minimal unital cyclic C_∞-algebras.

3. Classification of rational and real homotopy type of closed manifolds.

4. Crowley-Nordström's and Cavalcanti formality theorem revisited.

5. Final remarks

1. Models of RHT and real homotopy theory.

• (1975-1977) Sullivan model. The punchline: Replace a topological space X with a free commutative DGA. RHT of a topological space X is defined by $A_{PL}(X)$. The real homotopy type of M is defined by $\Omega^*(M)$.

• In Sullivan minimal model $(\Lambda V, d) \simeq A_{PL}(X)$ d encodes the k-invariant and Whitehead products.

• (1979) Halperin-Stasheff model $(\Lambda Z, D) \simeq (\mathcal{A}^*, d)$ is a free filtered DGCA, where $(\Lambda Z, d) \to H^*(X)$ is a Tate-Jozefiak resolution and $D = d + p$ is a perturbation of d s.t. $D^2 = 0$. The Punchline: Formality is not a binary; it's a filtration.

• (1969) Quillen Model: represented by DGLA (L, ∂) where $H_n(L, \partial) \cong \pi_{n+1}(X) \otimes \mathbb{Q}$. Here $L = \mathbb{L}(s^{-1}\tilde{H}_*(X, \mathbb{Q}))$ where the Lie bracket encodes the Whitehead products in the homotopy groups $\pi_*(X) \otimes \mathbb{Q}$.

• (2009) Kadeishvili minimal C_∞-algebra model $C_\infty(\mathcal{A}^*)$ $(H^*(\mathcal{A}^*), m) \overset{C_\infty\text{-quasi-isomorphic}}{\simeq} \mathcal{A}^*$. The structure is obtained by transferring the DGA data to its cohomology $m := (m_1 = 0, m_2 = m_2(H^*), m_3, m_4, \ldots)$. The punchline: All higher homotopy information (Massey product, k-invariants) is packed into the higher operation m_k on the finite dimensional space $H^*(\mathcal{A}^*)$.

2. Hodge homotopy and minimal unital cyclic C_∞-algebras. A Poincaré DGCA $(\mathcal{A}^*, d)$ of degree n over a field $\mathbb{K}$ is a DGCA $\mathcal{A}^* = \oplus_{k=0}^n \mathcal{A}^k$ s.t. $\dim H^*(\mathcal{A}) < \infty$ and $\exists$ a fundamental class $\int \in (H^n(\mathcal{A}))^\vee$ whose induced pairing

$$\langle \alpha^k, \beta^l \rangle = \begin{cases} \int \alpha^k \cdot \beta^l & \text{if } k+l=n, \\ 0 & \text{else,} \end{cases} \tag{1}$$

is non-degenerate. The pairing on $H^*(\mathcal{A})$ induces a pairing on $\mathcal{A}^*$:

$$\langle \alpha^k, \beta^l \rangle = \begin{cases} \int [\alpha^k \cdot \beta^l] & \text{if } k+l=n, \\ 0 & \text{else.} \end{cases} \tag{2}$$

- (FKLS2021/FL2023) Let $(\mathcal{A}^*, d, \cdot, \langle -,-\rangle)$ be a PDGCA. A Hodge homotopy for $(\mathcal{A}^*, d, \cdot, \langle -,-\rangle)$ is an operator $d^-\colon \mathcal{A}^* \to \mathcal{A}^{*-1}$ such that

$$d^-d^- = 0; \quad d^-dd^- = d^-; \quad dd^-d = d;$$

$$\langle \mathrm{Im}(d^-), \mathrm{Im}(d^-)\rangle = 0; \quad \langle \mathrm{Im}(\pi_{\mathcal{H}^*}), \mathrm{Im}(d^-)\rangle = 0,$$

$$\pi_{\mathcal{H}^*} := \mathrm{id}_{\mathcal{A}^*} - [d, d^-].$$

- $\mathcal{H}^* := \mathrm{Im}(\pi_{\mathcal{H}^*})$ - the harmonic subspace for d^-.

$$\mathcal{A}^* = \mathcal{H}^* \oplus \underbrace{dd^-\mathcal{A}^* \oplus d^-d\mathcal{A}^*}_{\mathcal{L}^*_{\mathcal{A}}}.$$

• $j: (\mathcal{H}^*, d_{\mathcal{H}} = 0) \to (\mathcal{A}^*, d)$ is a quasi-isomorphism.

$$(\mathcal{A}^*, d) \underset{j}{\overset{\pi_{\mathcal{H}^*}}{\rightleftarrows}} (\mathcal{H}^*, d_{\mathcal{H}} = 0), \qquad j\pi_{\mathcal{H}^*} - id = dd^- + d^-d.$$

$$\langle x, y \rangle = \langle x, \pi_{\mathcal{H}} y \rangle. \tag{3}$$

$$\langle d^- x, y \rangle = (-1)^{|x|} \langle x, d^- y \rangle. \tag{4}$$

Theorem 1 The transferred minimal C_∞-algebra, denoted by $C_\infty(\mathcal{H}^*)$, is unital and cyclic:

$\langle m_k(x_1, \ldots, x_k), x_{k+1} \rangle = (-1)^k (-1)^\epsilon \langle m_k(x_2, \ldots, x_{k+1}),$

where $\epsilon = |x_1|(|x_2| + \cdots + |x_{k+1}|)$.

Lemma 1. Generalized Odd Symmetry Assume that $\mathcal{A}^*$ is $(r-1)$-connected, with r odd. Then for every $x, y, z \in \mathcal{H}^r$ we have

$$\widehat{m}_k(x, \ldots, x, y) = (-1)^{k-1}\widehat{m}_k(y, x, \ldots, x),$$
$$\widehat{m}_k(x, y, \ldots y, z) = (-1)^{k-1}\widehat{m}_k(z, y \ldots, y, x).$$

Lemma 2. Generalized Even Symmetry for $k = 3$ Assume that $\mathcal{A}^*$ is $(r-1)$-connected, with r even. Then for every $y \in \mathcal{H}^r$, for every homogeneous x, z we have

$$\widehat{m}_3(x, x, y) = -(-1)^{|x|}\widehat{m}_3(y, x, x),$$
$$\widehat{m}_3(x, y, z) = -(-1)^{|x||z|}\widehat{m}_3(z, y, x).$$

In particular, for every $x, y \in \mathcal{H}^r$, we have

$$\widehat{m}_3(x, y, x) = 0.$$

Lemma 3. Generalized Even Symmetry for $k = 4$ Assume that $\mathcal{A}^*$ is $(r-1)$-connected, with r even. Then for every $x, y, z \in \mathcal{H}^r$ we have

$$\widehat{m}_4(x, x, x, y) = -\widehat{m}_4(y, x, x, x),$$
$$\widehat{m}_4(x, y, y, z) = -\widehat{m}_4(z, y, y, x).$$

- Example of a PDGCA over $\mathbf{R}$ admitting a Hodge homotopy is $(\Omega^*(M), d)$ with the harmonic subspace being the space of harmonic forms on M.

- Example od a PDGCA over $\mathbf{Q}$ admitting a Hodge homotopy is the finite dimensional Poincaré duality algebra which is weakly equivalent to a given simply connected PDGCA over $\mathbf{Q}$ by Lambrect-Stanley (2004). By Cieliebak-Fukaya-Latschev (2020) such a finite dimensional Poincaré duality algebra admits a Hodge homotopy.

3. Classification of RHT and real homotopy type of closed smooth manifolds

• The core innovation- isotopy modulo k. Purpose: Define the "resolution" at which you are looking at the homotopy type.

• **Definition** (Equality Modulo k): Two C_∞-structures are equal modulo k, $\mathcal{A}_1 = A_2 \pmod{k}$, if their operations m_i match for all $i \leq k$.

• **Definition** (Isotopy Modulo k): $A_1 \sim_k A_2$ if there is a C_∞-isotopy $\phi = (\mathrm{Id}, \phi_2, \dots)$ such that $\phi(A_1) = A_2 \pmod{k}$.

• The DGLA Perspective: We use the Arity Filtration on the Gerstenhaber DGLA: $\mathfrak{g}_{H^*} = F^1 \supset F^2 \supset \ldots$

• Isotopy modulo k is precisely the Gauge Action of the pronilpotent group on the quotient $\mathfrak{g}_{H^*}/F^{k+1}$.

The Big Idea: Two spaces are C_∞-isomorphic if and only if they are isotopic modulo k for all k.

Obstruction Theory: Level 3 and Level 4. Purpose: Show exactly where the "formality" breaks down.

• **The Primary Obstruction** ($k = 3$): The first non-trivial comparison happens at m_3.
$A_1 \sim_3 A_2 \iff [m_3^{(1)}] = [m_3^{(2)}] \in \mathsf{Harr}^{3,-1}(H^*, H^*)$.

• Cyclic Case: This lives in the cyclic cohomology $HC^{4,-1}(H^*)$.

- **The Secondary Obstruction** ($k = 4$):If m_3 is matched, the next barrier is an affine space $\mathcal{K}_4$. The obstruction class $[\tilde{\kappa}_4]$ depends on the choice of ϕ_2 used to fix m_3.

- **Key Theorem**: We provide a general Additivity Formula for higher obstructions $\mathcal{K}_k$ in any filtered DGLA.

The Classification Theorem for Manifolds

• Purpose: The "Killer App" — classifying homotopy types with finite data.

• **Theorem (Finite Classification)**: For a closed $(r-1)$-connected manifold M of dimension $n \leq l(r-1)+2$, the rational/real homotopy type is determined uniquely by the cohomology $H^*(M)$ and the isotopy modulo $(l-2)$.

- $n \leq 4r-2 \implies$ Formal (all $m_k \geq 3$ vanish).

- $n \leq 5r-3 \implies$ Homotopy type defined solely by the Primary Obstruction $[Tm_3]$.

- $n \leq 6r-4 \implies$ Determined by m_3 and the secondary obstruction m_4.

- **Advantage**: This turns the infinite Sullivan model into a finite set of cohomology classes.

4. Crowley-Nordström's and Cavalcanti's formality theorems revisited. (In collaboration with Domenico Fiorenza)

• (Crowley-Nordström 2020, Cavalcanti 2006 weaker version) Let $\mathcal{A}^*$ be $(r-1)$-connected Poincaré DGCA over $\mathbf{F} = \mathbf{R}, \mathbf{Q}$ with degree $n = 4r - 1$ and $b^r \leq 3$. If there is an element $\varphi \in H^{2r-1}$ such that $\varphi \cdot - : H^r \to H^{3r-1}$ is an isomorphism then $\mathcal{A}^*$ is intrinsically formal.

Remark Crowley-Nordström and Cavalcanti used Bianchi-Massey tensor and Sullivan theory, respectively.

Fiorenza-L. new proof (2026).

• Let $\psi : H^{3r-1} \to H^r$ - the inverse of $\varphi \cdot -$.

We also define the tensor

$$R\colon (H^r)^{\otimes 3} \to H^r,$$
$$x \otimes y \otimes z \mapsto \psi\Big(m_3(x, z, y)\Big),$$

Lemma 4 (F.-L. 2026) [Symmetries of R]

(a) Assume r is even. Then the tensor R satisfies:

1. $R_{ijkl} = -R_{jikl}$ (antisymmetry in the first two indices)

2. $R_{ijkl} = -R_{ijlk}$ (antisymmetry in the last two indices)

3. $R_{ijkl} = R_{klij}$ (pair swapping symmetry)

Therefore, R defines a linear map $S^2\wedge^2 H^r \to \mathbb{K}$. If $\dim H^r \leq 3$, then R also satisfies the Bianchi identity and so it is is an algebraic curvature tensor.

(b) If r is odd, we have similar symmetries. Hence R defines a linear map $S^2S^2H^r \to \mathbf{R}$.

• The first Bianchi Identity $(R(x,y)z+R(y,z)x+R(z,x)y=0)$ for r even isn't just a geometric miracle; it is the "geometric shadow" of the C_∞-relations.

Outline of the new proof.

1. r even. Lemma 4 says that in the case $b_r \leq 3$, the map R is defined uniquely by its Ricci tensor

$$Ric := \mathrm{tr}(R) : H^r \times H^r \to \mathbf{R}.$$

Then $Ric = R_0 + Ric^0$ (scalar curvature + traceless part of Ric). Using this decomposition and generalized symmetry Lemmas 1,2,3, we can find a Harrison cochain $\phi : S^2 H^* \to H^{*-1}$ such that

$$d\phi = m_3.$$

Since $\dim \mathcal{A}* = 4k - 1$, we conclude that $(\mathcal{A}^*, d)$ is formal.

2. r odd: two proofs : a) finding ϕ by solving a system of linear equations, which can be solved thanks to a lot of symmetries, b) Use the technique of finding a finite dimensional nondegenerate PDGCA $\mathcal{Q}_{\mathsf{small}}(\mathcal{A}^*)$ equivalent to a given 1-connected PDGCA $\mathcal{A}^*$ in Fiorenza-Kawai-Lê-Schwachhöfer 2026 to conclude that $\mathcal{Q}_{\mathsf{small}}(\mathcal{A}^*)$ is formal.

3. An extension of Cavalcanti's formality result

Theorem 3 (L.2026) Assume that $(\mathcal{A}^*, d^-)$ is a $(r-1)$-connected PDGCA of degree $n \leq 5r-2$ admitting Hodge d^- with $b^r \leq 2$.

1) If $\exists\, \varphi \in H^{n-2r}$ s.t. $\varphi \cdot - : H^r \to H^{n-r}$ is an isomorphism, then $\mathcal{A}^*$ is C_∞-quasi-isomorphic to a minimal unital cyclic C_∞-algebra whose multiplication m_k vanishes for $k \geq 4$.

2) If, moreover, $n \leq 4r$ then $\mathcal{A}^*$ is formal.

Remark. Theorem 2 (2) was obtained by Cavalcanti in 2006 for the case of compact smooth manifolds.

Outline of the proof. In Fiorenza-Kawai-Le-Schwachhöfer (2021) we prove Theorem 2 under the condition that $n \leq 5r - 3$ but without the Leftschetz condition. Thus we need to analyze what happens with $n = 5r-2$ under the Leftschetz condition. Using generalized symmetry, cyclicity, and unitality, we are able to show that $m_4 = 0$.

For the proof of Theorem 3(2), we need to find a Harrison coboundary ϕ such that $m_3 = d\phi$.

Then apply the finiteness theorem to conclude that if $m_4 = 0$, $[m_3] = 0 \in Harr^{3,-1}(H^*, H^*)$ then $\mathcal{A}^*$ is formal.

Final remarks

• Classical Rational Homotopy Theory often feels like an infinite-dimensional 'black box.' In my latest work, I provide a way to open that box by "stratifying" a manifold's homotopy type. I introduce the concept of "isotopy modulo k,' which acts like a Taylor expansion for the homotopy DNA of a space. Instead of solving for global isomorphisms, we analyze the structure level-by-level through a filtered DGLA.

2) The Harrison cohomology class $[m_3]$ corresponds to the Bianchi–Massey tensor of Crowley–Nordström (FL2025). "The primary obstruction $[m_3]$ relates to the Bianchi–Massey tensor. Our collaborative work (FL2026) shows that for specific connectivity and dimension, the symmetries of the resulting R-tensor force the vanishing of these obstructions. It remains an open problem to express the pentagon Massey tensor of Nagy–Nordström in terms of our higher obstruction classes.

3) Our recursive approach via affine obstruction spaces provides a computable alternative to the classical moduli space description V/G by Schlessinger–Stasheff. While their work characterizes the moduli of homotopy types as a global quotient of an algebraic variety, we analyze the equivalence problem level-by-level through the filtered structure of the Gerstenhaber DGLA, allowing for explicit calculations at each level.

4) There is a significant conceptual overlap between our unital cyclic models and the equivariant string topology framework of Cieliebak, Hajek, and Volkov. Our stratification provides the discrete algebraic foundation for the determination of IBL_∞-structures, linking the isotopy modulo k of C_∞-algebras to the analytic propagators used in configuration space integrals.

5) Based on the vanishing of secondary obstructions in the low-dimensional cases studied, we conjecture that Theorem can be strengthened, extending the dimension bound from $n \leq 5r-2$ to the critical threshold of $n \leq 6r-4$.

6) Finally, the classification of minimal C_∞-algebra enhancements on $H^*(X)$ is deeply linked to the L_∞-algebra of the automorphism group of X. Following the works by Berglund and Buijs-Félix-Murillo on the L_∞ models of mapping spaces, we intend to explore the relationship between our Harrison obstruction classes and the rational homotopy type of $Map(X, X)$ in future work.

https://arxiv.org/abs/2603.01219

THANK YOU FOR YOUR ATTENTION!