Legendrian non-isotopic unit conormal bundles in high dimensions

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§1. A question in contact topology & Main results

§2. A coproduct from string topology

§3. Outline of proof of Thm. 1

§1. Terminology

M : $(2n-1)$-dim. C^∞ manifold.

Def. $\alpha \in \Omega^1(M)$ is a <u>contact form</u>

$\overset{def}{\Longleftrightarrow}$ $\alpha \wedge \underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{n-1} \in \Omega^{2n-1}(M)$

is nowhere vanishing

Today, we only consider the following contact manifold.

Example $\mathbb{R}^n \ni (q_1, \cdots, q_n) \rightsquigarrow T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \ni (q_1, \cdots, q_n, p_1, \cdots, p_n)$

$S^*\mathbb{R}^n := \{ (q, p) \in T^*\mathbb{R}^n \mid p_1^2 + \cdots + p_n^2 = 1 \}$. sphere cotangent bundle

$$\alpha := \left(\sum_{i=1}^n p_i \, dq_i \right) \Big|_{S^*\mathbb{R}^n} \in \Omega^1(S^*\mathbb{R}^n)$$

$\Lambda \subset M$: C^∞ submanifold, $\dim \Lambda = n-1$.

Def. Λ is a Legendrian submanifold of (M, α)

$\overset{\text{def}}{\iff}$ $\alpha|_\Lambda = 0 \in \Omega^1(\Lambda)$.

Λ_0, Λ_1 : Legendrian submanifold of (M, α)

Λ_0 is Legendrian isotopic to Λ_1 ($\Lambda_0 \sim_{\text{Leg}} \Lambda_1$)

$\overset{\text{def}}{\iff}$ ${}^\exists(\Lambda_t)_{t \in [0,1]}$: smooth family of Legendrian submanifolds of (M, α).

- Classifying Legendrian submanifolds up to $\sim_{\text{Leg}}$ is an important problem in Contact topology.

Example & More specific question

<u>Example</u> $^\forall K \subset \mathbb{R}^n$: compact C^∞ submanifold

$$\Lambda_K := \{ (q,p) \in S^*\mathbb{R}^n \mid q \in K, \quad p \in (T_q K)^\perp \}$$

is a Legendrian submanifold of $(S^*\mathbb{R}^n, \alpha)$,

called the <u>unit conormal bundle</u> of K.

Obviously,

$$K_0 \sim_{C^\infty} K_1 \text{ in } \mathbb{R}^n \Rightarrow \Lambda_{K_0} \sim_{Leg} \Lambda_{K_1} \text{ in } (S^*\mathbb{R}^n, \alpha)$$

<u>Question</u> Does ($\Leftarrow$) hold ?

Preceding researches

- Thm (Shende '19, Ekholm-Ng-Shende '18)

  $K_0, K_1 \subset \mathbb{R}^3$: (smooth) knots

  $\Lambda_{K_0} \sim_{Leg} \Lambda_{K_1} \Rightarrow K_0 \sim_{C^\infty} K_1$ or $\mathrm{mirror}(K_1)$

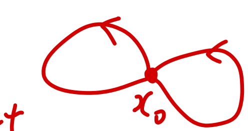
Tools: $\begin{cases} \text{Shende : microlocal sheaf theory} \\ \text{Ekholm-Ng-Shende: Legendrian contact homology} \end{cases}$

- Including high dimensions: $K_0, K_1 \subset \mathbb{R}^n$

$$\Lambda_{K_0} \sim_{Leg} \Lambda_{K_1} \Rightarrow \begin{cases} \mathbb{Z}[\pi_1(\mathbb{R}^n \setminus K_0)] \overset{\text{Shende '19}}{\cong} \mathbb{Z}[\pi_1(\mathbb{R}^n \setminus K_1)] \\ H_*(\underbrace{\Omega_{x_0}(S^n \setminus K_0)}_{\text{based loop space}}) \overset{\text{Asplund '21}}{\cong} H_*(\Omega_{x_0}(S^n \setminus K_1)) \end{cases}$$

$(S^n = \mathbb{R}^n \sqcup \{x_0\})$
$\quad\;\; \cup$
$\quad K_0, K_1$

preserving Pontrjagin product



Main results of O'25

For $K \subset \mathbb{R}^n$: compact submanifold

diagonal subset $\{(q,q) \mid q \in K\}$

$$\mathbb{H}_*(K) := H_{*+2-\mathrm{codim} K}(K \times K, \Delta_K ; \mathbb{F}_2).$$

In §2, I will introduce a coproduct

$$\delta_K : \mathbb{H}_{*+1}(K) \longrightarrow \mathbb{H}_*(K) \otimes \mathbb{H}_*(K).$$

Thm. 1

$\left\{ \begin{array}{l} \cdot\ \Lambda_{K_0} \sim_{Leg} \Lambda_{K_1}, \\ \cdot\ K_i \text{ is connected and } \mathrm{codim} K_i \geq 4 \quad (i=0,1) \end{array} \right.$

$\Rightarrow$ $\exists \varphi : \mathbb{H}_*(K_0) \xrightarrow{\cong} \mathbb{H}_*(K_1)$ preserving the coproducts

i.e.

$$\begin{tikzcd} \mathbb{H}_{*+1}(K_0) \arrow[r, "\delta_{K_0}"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{H}_*(K_0) \otimes \mathbb{H}_*(K_0) \arrow[d, "\varphi \otimes \varphi"] \\ \mathbb{H}_{*+1}(K_1) \arrow[r, "\delta_{K_1}"] & \mathbb{H}_*(K_1) \otimes \mathbb{H}_*(K_1) \end{tikzcd}$$

Application.

Thm. 2 $\forall \begin{cases} d \in \mathbb{Z}_{\geq 4} \\ \Sigma : \text{closed surface, connected, } \neq S^2 \end{cases}$

This Σ can be generalized.

$\exists f_0, f_1 : S^{d-1} \times \Sigma \hookrightarrow \mathbb{R}^{2d+1}$: embeddings, codim $= d$

s.t. for $K_0 := \mathrm{Im} f_0$ and $K_1 := \mathrm{Im} f_1$,

(i) Λ_{K_0} and Λ_{K_1} have the same

$\underbrace{\text{"classical invariants"}}_{\text{Legendrian isotopy}}$ $\begin{cases} \text{Thurston-Bennequin number} \\ \text{Legendrian regular homotopy class} \\ C^\infty \text{ isotopy class} \end{cases}$

(ii) $\delta_{K_0} = 0$, $\delta_{K_1} \neq 0$.

In particular, $\Lambda_{K_0} \not\simeq_{Leg} \Lambda_{K_1}$ by Thm. 1.

§2. Coproduct δ_K from string topology

$K \subset \mathbb{R}^n$: compact submanifold

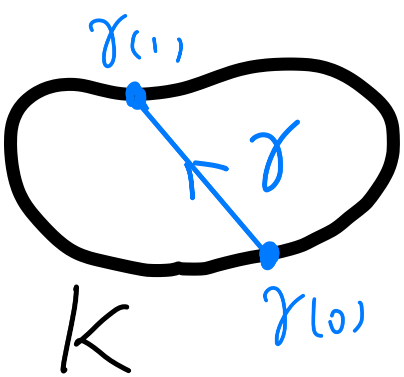
$$H_*(K\times K, \Delta_K ; \mathbb{F}_2) \xrightarrow{\delta_K} H_{*+1-\mathrm{codim}K}\big((K\times K, \Delta_K)^{\times 2} ; \mathbb{F}_2\big)$$

$$\cong \text{ Künneth}$$

$$H_*(K\times K, \Delta_K ; \mathbb{F}_2)^{\otimes 2}\,[1-\mathrm{codim}K]$$

We have an identification

$$\begin{array}{ccc} \{\gamma : [0,1]\to\mathbb{R}^n : \text{linear} \mid \gamma(0),\gamma(1)\in K\} & \xrightarrow{\cong} & K\times K \\ \gamma & \longmapsto & (\gamma(0),\gamma(1)) \\ \cup & & \cup \\ \{\text{constant paths in } K\} & & \Delta_K \end{array}$$



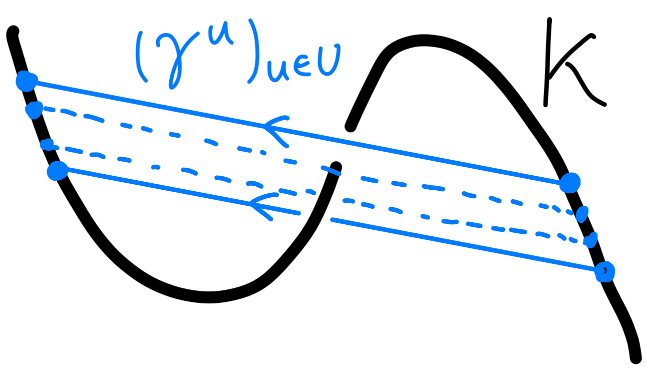
$$P_K^{lin} := \{\gamma : [0,1] \to \mathbb{R}^n : \text{linear} \mid \gamma(0), \gamma(1) \in K\} \quad (\cong K \times K)$$

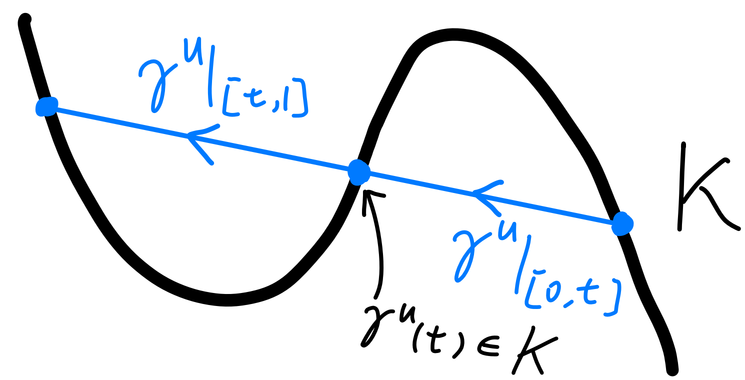
<u>Idea</u> $(\gamma^u)_{u \in U}$: a family (or chain) of $\gamma^u \in P_K^{lin}$ parametrized by U

$\rightsquigarrow$ $\left(\gamma^u|_{[0,t]}, \gamma^u|_{[t,1]}\right)_{(t,u) \in \tilde{U}}$

: a family (or chain) in $P_K^{lin} \times P_K^{lin}$ parametrized by

$$\tilde{U} := \left\{(t,u) \;\middle|\; \begin{array}{l} 0 < t < 1, \; u \in U \\ \gamma^u(t) \in K \end{array}\right\}.$$

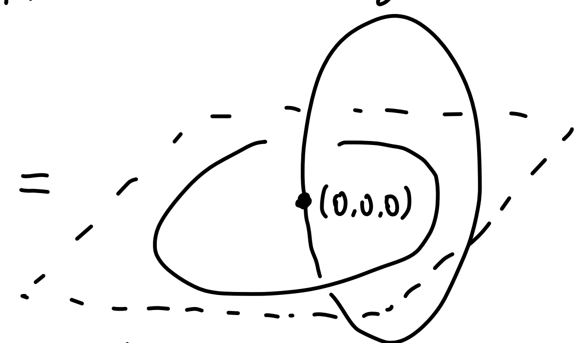




$$\begin{array}{ccc} \delta_K : H_p\left(P_K^{lin}, \left\{\begin{smallmatrix}\text{const.}\\ \text{paths}\end{smallmatrix}\right\}\right) & \longrightarrow & H_{p+1-\operatorname{codim}K}\left(\left(P_K^{lin}, \left\{\begin{smallmatrix}\text{const.}\\ \text{paths}\end{smallmatrix}\right\}\right)^{\times 2}\right) \\ \cong & & \cong \\ HM_p(K\times K, \Delta_K) & & HM_{p+1-\operatorname{codim}K}\left((K\times K, \Delta_K)^{\times 2}\right) \end{array}$$

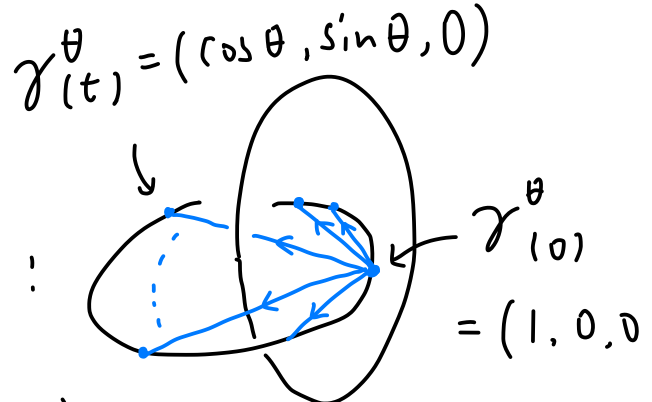
Remark

A precise definition can be given by using Morse homology of $K\times K$.

Example $K =$  $\subset \mathbb{R}^3$

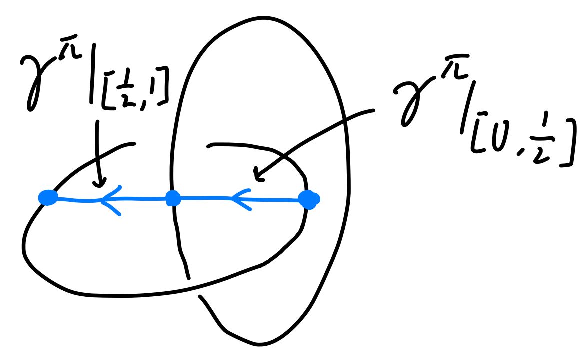
$x \in H_1\left(P_K^{lin}, \left\{\begin{smallmatrix}\text{const.}\\ \text{paths}\end{smallmatrix}\right\}\right)$

represented by $(\gamma^\theta)_{\theta\in[0,2\pi]}$:



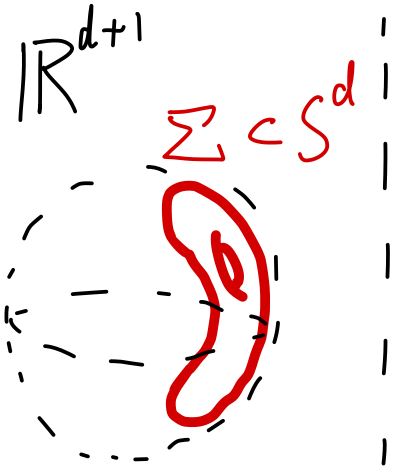
$\rightsquigarrow \delta_K(x) \in H_0\left(\left(P_K^{lin}, \left\{\begin{smallmatrix}\text{const.}\\ \text{paths}\end{smallmatrix}\right\}\right)^{\times 2}\right)$

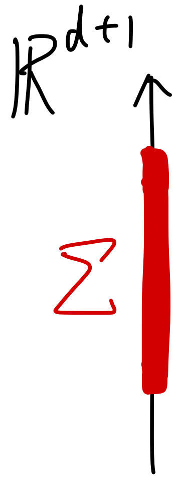
represented by a pair of paths :



Examples K_0 and K_1 of Thm. 2

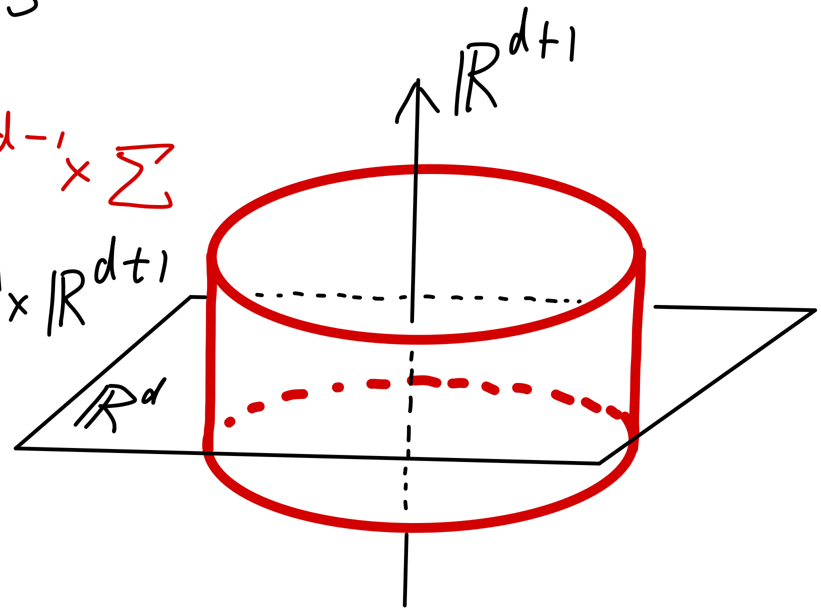
$d \geq 4$, Σ : closed surface $\neq S^2$



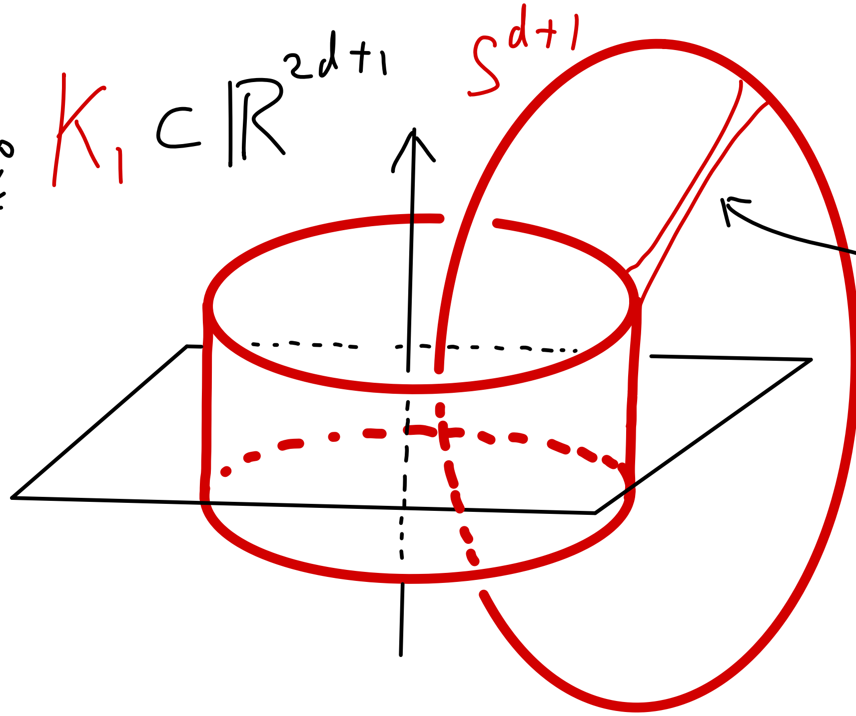


$K_0 := S^{d-1} \times \Sigma$

$\subset \mathbb{R}^d \times \mathbb{R}^{d+1}$



$K_0 = S^{d-1} \times \Sigma \overset{\text{diffeo}}{\cong} K_1 \subset \mathbb{R}^{2d+1}$



(embeded) Connected sum.

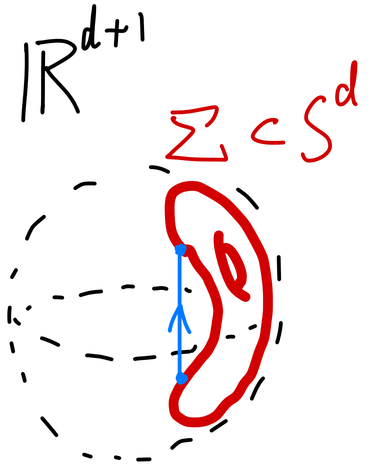
claim. $\delta_{K_0}(x) = 0 \quad {}^\forall x \in H_*(P^{lin}_{K_0}, \{\text{const. paths}\})$

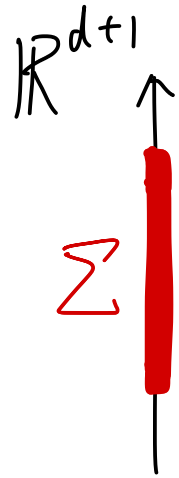
∵ ${}^\forall \gamma \in P^{lin}_{K_0}$

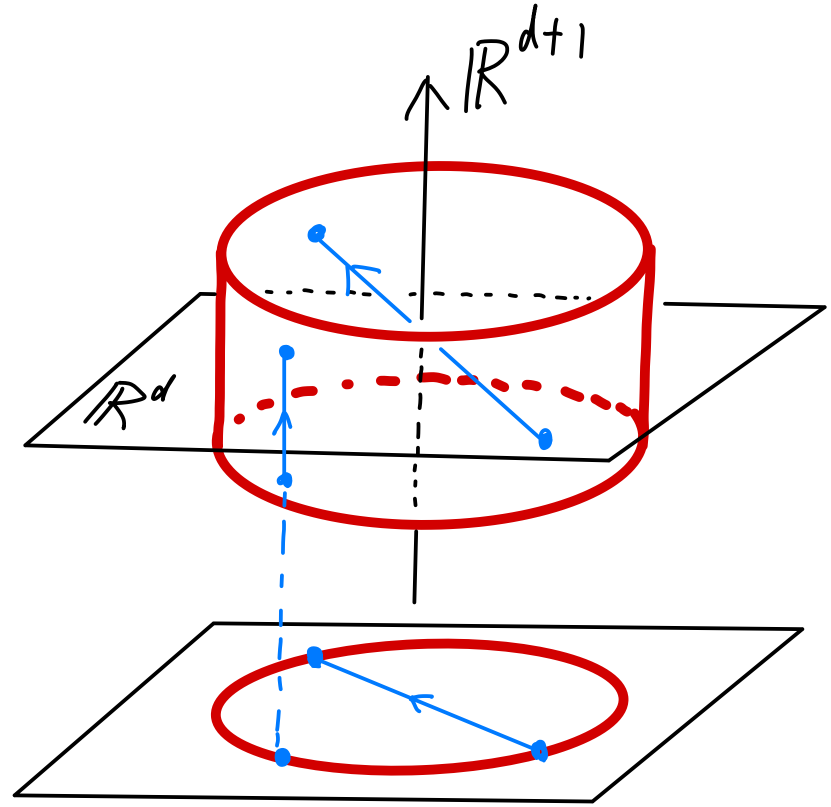
$0 < {}^\forall t < 1$

$\gamma(t) \notin K_0$

□







claim. $\delta_{K_1}({}^\exists x) \neq 0$

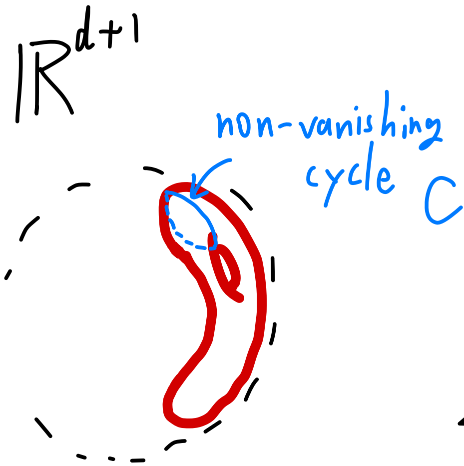
∵ $H_1(\underset{\not\cong\, S^2}{\Sigma}; \mathbb{F}_2) \neq 0$.

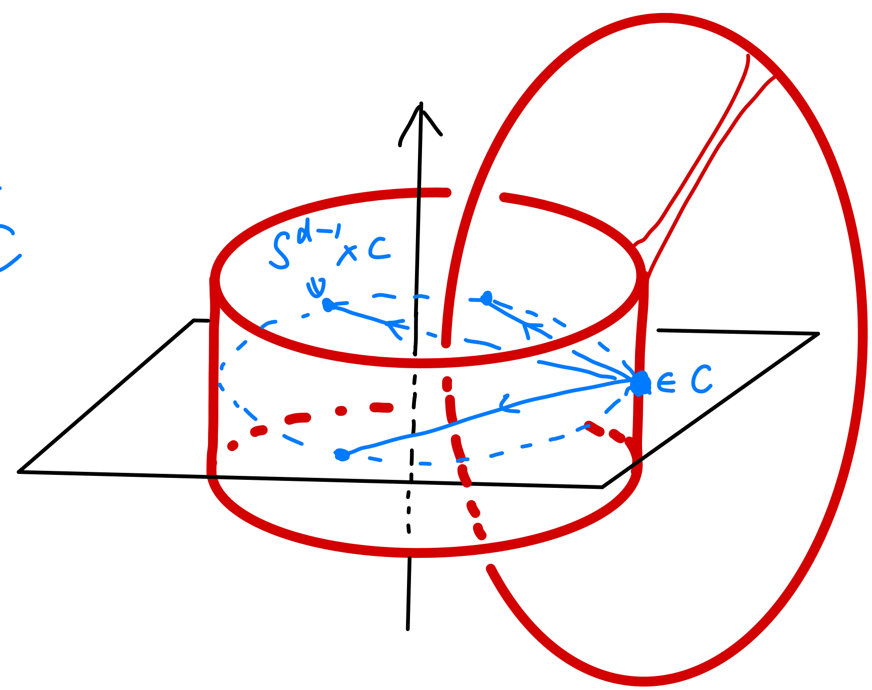
${}^\exists x$ represented by

$(\gamma^u)_{u \in C \times (S^{d-1} \times C)}$

s.t. $\delta_{K_1}(x) \neq 0$.

□





§3. Outline of proof of Thm. 1

(i) We use a Legendrian isotopy invariant

$$(HL_*(\Lambda) , \delta_\Lambda)$$

Strip Legendrian Contact homology   coproduct.

defined under index-positivity condition on Λ

(ii) If $K \subset \mathbb{R}^n$: compact, connected submanifold

$\operatorname{codim} K \geq 4$,

then $\Lambda_K \subset S^*\mathbb{R}^n$ is index-positive.

unit conormal bundle of K

Moreover,

$$(HL_*(\Lambda_K) , \delta_{\Lambda_K}) \cong (\mathbb{H}_*(K) , \delta_K)$$

$$\mathbb{H}_*(K) = H_{*+2-\operatorname{codim} K}(K \times K, \Delta_K)$$

(Proof of $\underline{\text{Thm. 1}}$ assuming (i), (ii))

$$\begin{cases} \Lambda_{K_0} \sim_{Leg} \Lambda_{K_1} & \text{in } S^*\mathbb{R}^n \\ K_i \text{ is connected and } \operatorname{codim} K_i \geq 4 & (i = 0, 1) \end{cases}$$

$$\Rightarrow (\mathbb{H}_*(K_0), \delta_{K_0}) \overset{\text{(ii)}}{\cong} (HL_*(\Lambda_{K_0}), \delta_{\Lambda_{K_0}})$$

$$\overset{\text{(i)}}{\cong} (HL_*(\Lambda_{K_1}), \delta_{\Lambda_{K_1}})$$

$$\overset{\text{(ii)}}{\cong} (\mathbb{H}_*(K_1), \delta_{K_1}) \qquad \square$$

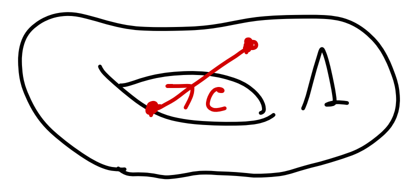
$$\left(\begin{array}{l} \text{For } i = 0, 1. \\ \mathbb{H}_*(K_i) := H_{*+2-\operatorname{codim} K_i}(K_i \times K_i, \Delta_{K_i} ; \mathbb{F}_2) \\ \qquad \delta_{K_i} : \text{the coproduct in §2.} \end{array} \right)$$

(i) Strip LCH and coproduct

Λ : compact, connected Legendrian submanifold of $(S^*\mathbb{R}^n, \alpha)$ ← generalized to contact manifold w/ no contractible periodic Reeb orbit.

s.t. Maslov class $= 0 \in H^1(\Lambda;\mathbb{Z})$

R_α : Reeb vector field associated w/ α (In fact, $R_\alpha = \sum_{i=1}^n p_i \partial_{q_i}$ on $S^*\mathbb{R}^n$)



$$\mathcal{R}(\Lambda) := \left\{ c : [0,T] \longrightarrow S^*\mathbb{R}^n \;\middle|\; \begin{array}{l} \dot{c}(t) = (R_\alpha)_{c(t)} \quad {}^\forall t \in [0,T] \\ c(0), c(T) \in \Lambda \end{array} \right\}$$

Reeb chord of Λ

For generic Λ, ${}^\forall c \in \mathcal{R}(\Lambda)$ is non-degenerate and we can define

$$\mu : \mathcal{R}(\Lambda) \longrightarrow \mathbb{Z} : c \mapsto \mu(c)$$: Maslov index

pseudo-holomorphic strips

J : almost complex str on $\mathbb{R}_r \times S^*\mathbb{R}^n$

s.t. $\begin{cases} \cdot \ \mathbb{R}\text{-invariant} \quad (\mathbb{R} \curvearrowright \mathbb{R} \times S^*\mathbb{R}^n : r \cdot (r', x) = (r + r', x)) \\ \cdot \ d(e^r \alpha)(\cdot, J \cdot) \text{ is a Riemannian metric on } \mathbb{R} \times S^*\mathbb{R}^n \\ \cdot \ J(\partial_r) = R_\alpha, \ J(\ker\alpha) \subset \ker\alpha \end{cases}$

$^\forall c_0, c_1 \in \mathcal{R}(\Lambda)$

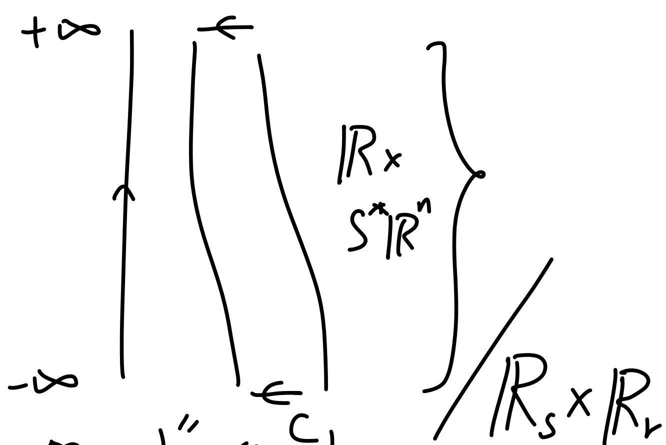
$$\mathcal{M}_{\Lambda, J}(c_0 ; c_1)$$

$$:= \{ u : \mathbb{R} \times [0,1] \ni (s,t) \longrightarrow \mathbb{R} \times S^*\mathbb{R}^n$$

$$\Big| \cdot \partial_s u + J \partial_t u = 0$$

$$\cdot \ u(\mathbb{R} \times \{0,1\}) \subset \mathbb{R} \times \Lambda, \quad \text{asymptotic condition} :$$

$c_0 = $ "$u(+\infty, \cdot)$"



"$u(-\infty, \cdot)$" $= c_1$

For generic Λ and J, $\mathcal{M}_{\Lambda,J}(c_0,c_1)$ is a C^∞ manifold of
dim $= \mu(c_0) - \mu(c_1) - 1$.

$$\begin{cases} C_*(\Lambda) := \displaystyle\bigoplus_{c \in R(\Lambda)} \mathbb{F}_2 \cdot c \qquad |c| := \mu(c) - 1. \\[2ex] d_J : C_{*+1}(\Lambda) \longrightarrow C_*(\Lambda) \\[1ex] \qquad c_0 \longmapsto \displaystyle\sum_{|c_1| = |c_0| - 1} \Big(\#_{\mathbb{F}_2} \underbrace{\mathcal{M}_{\Lambda,J}(c_0 ; c_1)}_{\text{0-dim. compact.}} \Big) \cdot c_1 \end{cases}$$

$d_J \circ d_J \neq 0$ in general. However,

Fact

$(\star)$ $c \in R(\Lambda)$ s.t. homotopic w/ endpoints in Λ to a constant path

$\Rightarrow \mu(c) > 1 \iff |c| > 0$.

$\Rightarrow d_J \circ d_J = 0$.

We obtain $HL_*(\Lambda) := H_*(C_*(\Lambda), d_J)$: strip Legendrian contact homology of Λ

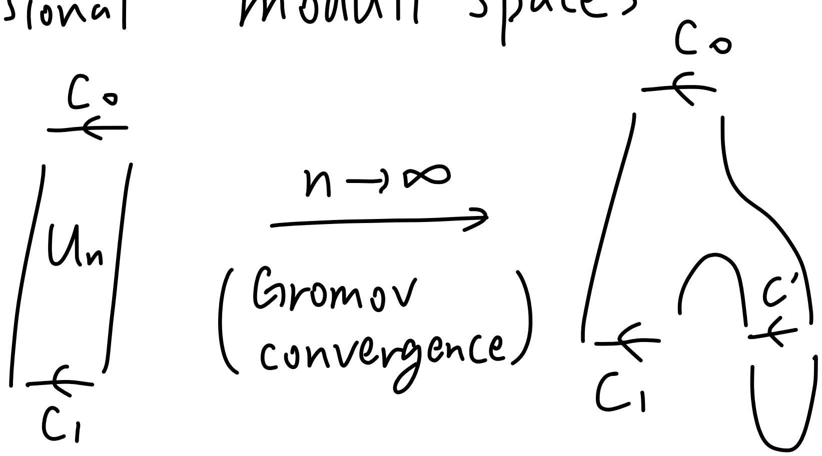
($\circledast$: $c \in R(\Lambda)$, null-homotopic $\Rightarrow \mu(c) > 1$.)

Remark $\circledast$ index positivity

- $\circledast$ prevents a "bubble" for pseudo-holomorphic strips in 1-dimensional moduli spaces

$\mathcal{M}_{\Lambda,J}(c_0;c_1) \ni$

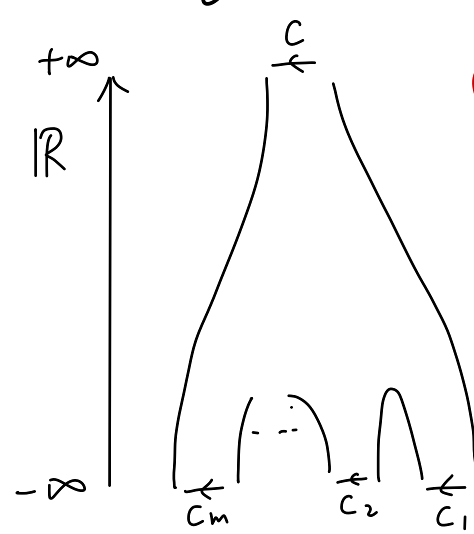
$|c_0| - |c_1| - 1 = 1$



- Without $\circledast$, Chekanov-Eliashberg DGA of Λ can be defined.

It involves pseudo-hol. curves

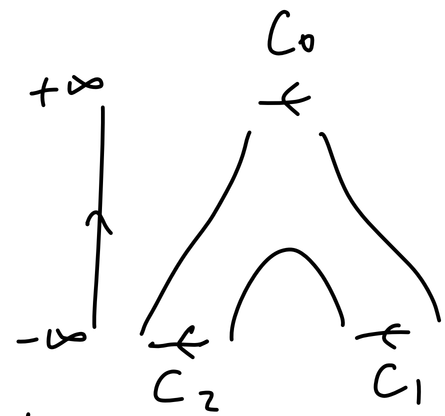
$\forall m \in \mathbb{Z}_{\geq 0}$



$\circledast \xrightarrow{\text{reduction}} (C_*(\Lambda), d_J)$

$$\delta_J : C_{*+1}(\Lambda) \longrightarrow C_*(\Lambda)^{\otimes 2}$$

is defined by using pseudo-hol. curves



With $\star$, δ_J is a chain map and induces

$$\delta_\Lambda : HL_{*+1}(\Lambda) \longrightarrow HL_*(\Lambda) \otimes HL_*(\Lambda)$$

Thm $\Lambda_0, \Lambda_1 \subset S^*\mathbb{R}^n$: compact. connected Legendrian submfd satisfying $\star$

$$\Lambda_0 \sim_{Leg} \Lambda_1 \Rightarrow HL_*(\Lambda_0) \overset{\exists}{\cong} HL_*(\Lambda_1)$$

preserving the coproducts.

(ii) $K \subset \mathbb{R}^n$: compact, connected submanifold

<u>Lem.</u> $^{\forall} c \in \mathcal{R}(\Lambda_K)$, $\mu(c) \geq \operatorname{codim} K - 2$

In particular, $\operatorname{codim} K \geq 4 \Rightarrow \Lambda_K$ satisfies $\circledast$

Thus, $(HL_*(\Lambda_K), \delta_{\Lambda_K})$ is defined.

<u>Thm</u> $HL_*(\Lambda_K) \overset{\exists}{\cong} H_{*+2-\operatorname{codim}K}(K \times K, \Delta_K)$ $\quad \uparrow \; H_0^{\text{string}}(K)$

which intertwines δ_{Λ_K} and δ_K $\quad$ δ_K : chain-level def.

The idea of proof is from <u>Cieliebak-Ekholm-Latschev-Ng '17</u>

on the Chekanov-Eliashberg DGA of Λ_K for $K \subset \mathbb{R}^3$: knot

Note: $\mathcal{P}_K := \{ \gamma : [0,1] \xrightarrow{C^\infty} \mathbb{R}^n \mid \gamma(0), \gamma(1) \in K \}$

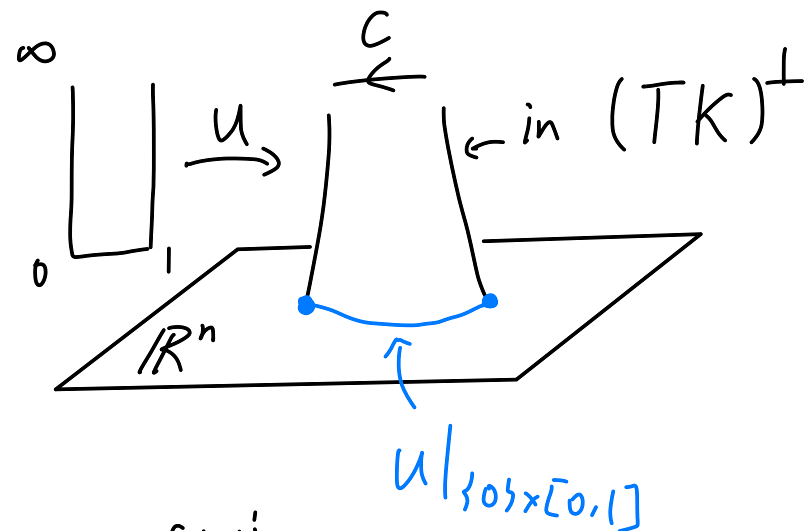
$\underset{\text{homotopy equivalent}}{\simeq} \mathcal{P}_K^{lin} = \{ \gamma : [0,1] \to \mathbb{R}^n : \text{linear} \mid \gamma(0), \gamma(1) \in K \} \cong K \times K$

$\gamma \mapsto (\gamma(0), \gamma(1))$

Sketch of $HL_*(\Lambda_K) \xrightarrow{\cong} H_{*+2-\mathrm{codim}K}(K \times K, \Delta_K)$

$\mathcal{N}(c)$: moduli space of pseudo-hol. curves in $T^*\mathbb{R}^n$

$$u : [0,\infty) \times [0,1] \longrightarrow T^*\mathbb{R}^n$$



$\overline{\mathcal{N}(c)}$: the Gromov compactification of $\mathcal{N}(c)$.

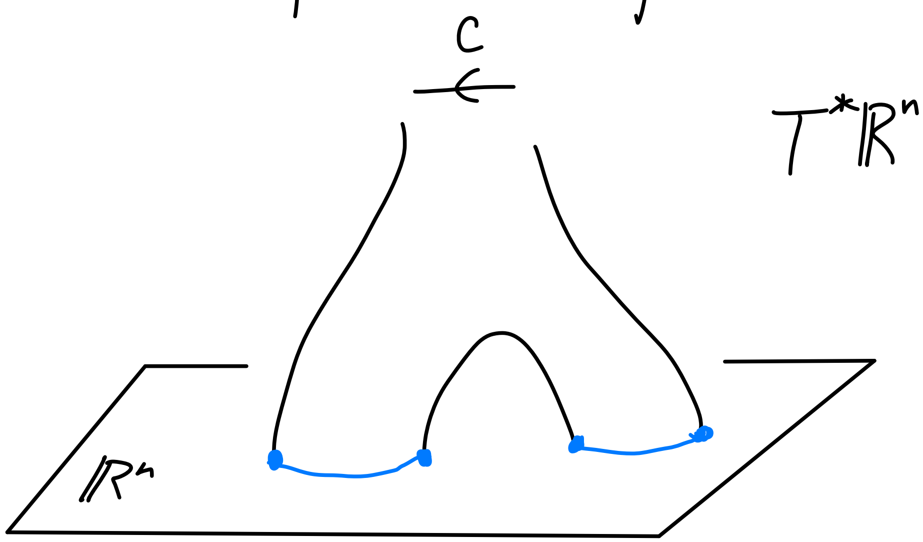
$$C_*(\Lambda_K) \longrightarrow C^{\mathrm{sing}}_{*+2-\mathrm{codim}K}\left(P_K, \begin{Bmatrix}\text{const.}\\ \text{paths}\end{Bmatrix}\right) \underset{\text{quasi-isom}}{\simeq} C^{\mathrm{sing}}(K \times K, \Delta_K)$$

$$c \longmapsto \left(\begin{matrix} \overline{\mathcal{N}(c)} \longrightarrow P_K \\ u \longmapsto u|_{\{0\}\times[0,1]} \end{matrix} \right)$$

Actually, I used a Morse chain complex of $K \times K$

Intertwing the coproducts

We use moduli space of pseudo-hol. curves



and observe the boundary of its compactification,

which contains