Category of finite sets and cohomology theories of commutative algebras

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Praha 10.12. 2025

Notations

K is a fixed commutative ring.

Mod the category of modules over K.

$\otimes = \otimes_K$ and $Hom = Hom_K$.

For a small category C, we let C-*Mod* be the category of all covariant functors $C \to Mod$. Similarly, *Mod*-C denotes the category of all contravariant functors $C \to Mod$.

Tensor product of functors

For a right C-module N and a left C-module M we let $N \otimes_{\mathsf{C}} M$ be the K-module generated by all elements $x \otimes y$, where $x \in N(c),\ y \in M(c)$ and $c \in \mathsf{C}$, modulo the relations

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$$

$$(ax) \otimes y = a(x \otimes y) = x \otimes (ay), \quad \alpha^*(x') \otimes y = x' \otimes \alpha_*(y).$$

Here $\alpha : c \to d$ is a morphism in C.

Categories related to finite sets

$\mathcal{F}$ = the category of all finite sets as objects and maps as morphisms. For any $n \geq 0$, we let $\underline{n}$ be the set $\{1, ..., n\}$. Hence $\underline{0}$ is the empty set. We will assume that the objects of $\mathcal{F}$ are the sets $\underline{n}, \ \ n \geq 0$.

$\mathcal{F}_0$ = the full subcategory of $\mathcal{F}$ all finite nonempty sets.

Γ = the category of all finite pointed sets and pointed maps.

Ω = the category of all finite sets as objects and surjective maps as morphisms.

Exponential left $\mathcal{F}$-modules

A functor $T : \mathcal{F} \to Mod$ is **exponential** if for any finite sets X and Y there is given an isomorphism

$$mu_{X,Y} : T(X \coprod Y) \to T(X) \otimes T(Y)$$

which is natural on X and Y and satisfy some coherent conditions (symmetric monoidal functor).

Main observation

If $T : \mathcal{F} \to Mod$ is **exponential** then $T(\underline{n}) = A^{\otimes n}$, where $A = T(\underline{1})$.

The category of commutative algebras is equivalent to the category of covariant exponential functors $\mathcal{F} \to Mod$ and the category of contravariant exponential functors is equivalent to the category of cocommutative (and coassociative) coalgebras.

From commutative algebras to exponential functors

If A is a commutative algebra, then the functor $\mathcal{L}_*(A) : \mathcal{F} \to Mod$ is exponential, where

$$\mathcal{L}_*(A)(\underline{n}) = A^{\otimes n}.$$

For any map $f : \underline{n} \to \underline{m}$, the action of f on $\mathcal{L}_*(A)$ is given by

$$f_*(a_1 \otimes \cdots \otimes a_n) := b_1 \otimes \cdots \otimes b_m,$$

where

$$b_j = \prod_{f(i)=j} a_i, \; j = 1, \cdots, n.$$

From exponential functors to commutative algebras

Conversely, assume T is an exponential functor. We let A be the value of T on $\underline{1}$. The unique map $\underline{2} \to \underline{1}$ yields a homomorphism

$$\mu : A \otimes A \cong T(\underline{2}) \to T(\underline{1}) = A.$$

On the other hand the unique map $\underline{0} \to \underline{1}$ yields a homomorphism $\eta : K = T(\underline{0}) \to T(\underline{1}) = A$. The pair (μ, η) defines on A a structure of commutative and associative algebra with unit.

Main question

We have seen that the category of commutative algebras can be seen as a subcategory of left $\mathcal{F}$-modules. Hence one can think on left $\mathcal{F}$-modules as generalised algebras.

What constructions and notions of commutative algebras have extensions to left $\mathcal{F}$-module?

De Rham cohomology, Hochschild homology, cyclic homology, (topological) Andre-Quillen homology have such extensions. See

- J.-L. Loday. Opérations sur l'homologie cyclique des algèbres commutatives. Invent. Math. 96, No. 1, 205-230 (1989))
- T. Pirashvili. André-Quillen homology via functor homology. Proc. Am. Math. Soc. 131, No. 6, 1687-1694 (2003)
- T. Pirashvili, B. Richter. Robinson-Whitehouse complex and stable homotopy. Topology 39, No. 3, 525-530 (2000).

An interesting open problem is whether etale cohomology have such an extension.

Since the category of left $\mathcal{F}$-modules is abelian with enough projective and injective objects, one can use the methods of classical homological algebra to study corresponding problems on commutative algebras. I will explain this later in a concrete example.

The smallest simplicial model of S^1

Consider the following simplicial set. In dimension n it is given by the set

$$[n] = \{0, 1, \cdots, n\}$$

the face operations given by

$$d_i(j) = \begin{cases} j, & j \leq i \\ j-1, & j > i \end{cases} \quad s_i(j) = \begin{cases} j, & j < i \\ j+1, & j \geq i \end{cases}$$

One easily sees that it has only two nongendered simplex, one in the dimension 0 and one in the dimension one. Hence it is a simplicial model of S^1.

Hochschild homology of a left Γ-module

For any functor $F : \Gamma \to Mod$ we can apply F degreewise to obtain a simlicial module which we denote by $F(S^1)$. Now define the Hochschild homology of a functor F by

$$HH_*(F) = \pi_* F(S^1)).$$

Let A be a commutative algebra and M be an A-module. We have a well-defined functor

$$\mathcal{L}(A, M) : \Gamma \to Mod$$

for which

$$\{0, 1, ..., n\} \mapsto M \otimes A^{\otimes n}.$$

Comparing the definition we see that $H_*(A, M) = H_*(\mathcal{L}(A, M)$.

Higher order Hochschild homology

This observation suggest the following generalization of Hochschild homology. Fix a pointed simplicial set Y_* and for any functor $F : \Gamma \to Mod$ we can set:

$$H_*^{Y_*}(F) = \pi_* F(Y_*)).$$

Thus for $F = \mathcal{L}(A, M)$ and $Y_* = S^1$ we obtain the classical Hochschild homology $HH_*(A, M)$.

I investigated the case when S^1 is replaced by S^d. This is so called higher order Hochschild homology. However there are several other important cases, studied by other mathematicians, Namely, when Y is the wedges of S^1 or Y is the surface of genus g.

Fundamental spectral sequence

Let Y_* be a pointed simplicial set. It is well-known that the homology $H_*(Y_*)$ of Y_* is a cocommutative coaugmented coalgebra. Therefore one can consider the corresponding exponential contravariant functor

$$\mathcal{J}(H_*Y) : \mathcal{F}^{op} \to Mod$$

Since $H_*(Y_*)$ is graded, the functor $\mathcal{J}(H_*Y)$ is also graded.
Theorem 1. Let F be a left Γ-module and let Y_* be a pointed simplicial set. Then there exists a spectral sequence

$$E^2_{pq} = Tor^{\Gamma}_p(\mathcal{J}_q(H_*Y_*), F) \Longrightarrow \pi_{p+q}(F(Y_*)).$$

Independent on models

As an immediate consequence of SS we obtain:

If a simplicial map $Y \to Y'$ induces an isomorphism on homology, then $H_*^Y(F) \to H_*^{Y'}(F)$ is an isomorphism.

Thus higher order homology is independent on the combinatorial model of the space Y.

The case $Y = S^d$

Theorem 2. There exists a spectral sequence

$$E^2_{pq} \Longrightarrow \pi_{p+q} F(S^d), \; d \geq 1,$$

with $E^2_{pq} = 0$ if $q \neq dj$ and

$$E^2_{pq} = Tor^{\Gamma}_p(\Lambda^j \circ t, F) \text{ if } q = dj \text{ and } d \text{ is odd}$$
$$E^2_{pq} = Tor^{\Gamma}_p(\theta^j, F) \text{ if } q = dj \text{ and } d \text{ is even.}$$

Moreover, if K is a field of characteristic zero, then the spectral sequence degenerates:

$$\pi_n F(S^d) \cong \bigoplus_{p+dj=n} Tor^{\Gamma}_p(\Lambda^j \circ t, F)$$

if d is odd and

$$\pi_n F(S^d) \cong \bigoplus_{p+dj=n} Tor^{\Gamma}_p(\theta^j, F)$$

if d is even.

Hodge decomposition

Hodge decomposition is a main tool in complex geometry. It has a combinatorial analogue, known as Hodge decomposition of Hochschild homology, discovered by Quillen in 60's and also by Gerstenhaber-Shack and Loday. They proved that if A is commutative and M is symmetric, then there is a decomposition

$$HH_n(A, M) = HH_n^{(1)}(A, M) \oplus \cdots \oplus HH_n^{(n)}(A, M)$$

In dimension two, this decomposition is quite easy and it is based on symmetric and antisymmetric tensors

$$x \otimes y = \frac{x \otimes y + y \otimes x}{2} + \frac{x \otimes y - y \otimes x}{2}$$

so, they are induced by the following idempotents

$$e_2^1 = \frac{id + (12)}{2}, \quad e_2^1 = \frac{id - (12)}{2}$$

Gerstenhaber-Shack and Loday used special idempotents in the group algebra of symmetric groups known as Euler idempotents. For $n = 3$ they look as follows:

$$e_3^1 = \frac{1}{3}id - \frac{1}{6}((123) + (132) - (12) - (23)) - \frac{1}{3}(13),$$

$$e_3^2 = \frac{1}{2}(id + (13))$$

$$e_3^3 = \frac{1}{6}(id + (123) + (132) - (12) - (13) - (23))$$

and are more complicated for $n > 3$.

I proved that the decomposition appeared in Theorem 2 for $d = 1$ is isomorphic to Hodge decomposition for Hochschild homology.

$$HH_n(F) = \bigoplus_{i=1}^{n} HH_*^{(i)}(F)$$

Thus

$$HH_*^{(i)}(F) = Tor_*^{\Gamma}(\Lambda^i \circ t, F)$$

The groups $\pi_* F(S^d)$ and playing with Chinese puzzles

Let d be an odd number. Comparing above results one sees that in the characteristic zero case the groups

$$\pi_n F(S^d) \cong \bigoplus_{i+dj=n} H^{(j)}_{i+j}(F) \tag{1}$$

for different d differ only by the way of taking the pieces $H^{(i)}_n(F)$ in the decomposition (1).

The same remark is true also for even d. The knowledge of the decomposition for $d = 2$ completely determines the decomposition for all even dimensional spheres. However in even case, in the decomposition of $\pi_n F(S^d)$, only the group $Harr_{n-d+1}(F) \cong H^{(1)}_{n-d+1}(F)$ belongs to Loday's decomposition; all other groups are new.

An equivalence of categories

For any $n \geq 1$ and any i such that $1 \leq i \leq n$, one defines the pointed maps

$$r_i : [n] \to [n-1]$$

by $r_i(i) = 0$, $r_i(j) = j$ if $j < i$ and $r_i(j) = j - 1$, if $j > i$.

For a left Γ-module T one defines the functor

$$cr(T) : \Omega \to Vect$$

by

$$cr(T)(\underline{n}) := \bigcap_{i=1}^{n} ker\ (r_{i*} : T([n]) \to T([n-1])).$$

For a surjection $f : \underline{n} \to \underline{m}$ one denotes by $f_0 : [n] \to [m]$ the unique pointed map which extends f. Then the homomorphism $(f_0)_* : T([n]) \to T([m])$ maps $cr(T)(\underline{n}) \subset T([n])$ to $cr(T)(\underline{m})$.

Hence we really obtain a functor Γ-$Mod \to \Omega$=Mod**, which is an equivalence of categories**.

The functors θ^j and t

Clearly if M is a representation of Σ_n, then there exist a unique right Ω-module $\tilde{M}$ for which $\tilde{M}(\underline{k}) = 0$, if $k \neq 0$ and $\tilde{M}(\underline{n}) = M$ as Σ_n-modules.

Now take M to be one dimensional sign and trivial representations. The corresponding Γ-modules are $\Lambda^n \circ t$ and θ^n respectively.