Universal, almost universal and related spacetimes

V. Pravda, Institute of Mathematics, AS CR, Prague

May 27, 2020

Universal and almost universal:

VSI and math formalism - older works with:
A. Coley and R. Milson (Halifax, Canada),
H. Reall and M. Durkee (Cambridge, UK)
1 Background
   - Weyl tensor
   - Einstein’s equations

2 Algebraic classification of tensors

3 VSI spacetimes

4 Universal spacetimes
   - Simple example - quadratic gravity
   - Main results
   - Main points of the selected proofs
   - Generalized Ghanam-Thompson metric
   - Almost universal spacetimes

5 Conclusions
Prelude: first results in 4d: 1990s picture

- **pp-waves**: Spacetimes admitting a null covariantly constant vector $\ell$, $\nabla_a \ell_b = 0$.
- **VSI spacetimes**: spacetimes with vanishing curvature invariants ($R_{abcd}R^{abcd}$, $R_{abcd;e}R^{abcd;e}$ etc. vanish)
- **Universal spacetimes**: “vacuum solutions to all theories”

- pp-waves are VSI:
  

- pp-waves are universal:
  

$$pp \iff VSI \iff U$$
Decomposition of Riemann tensor - Weyl tensor

**trace: Ricci tensor**

\[ R_{ab} = R^{e}_{aeb}, \quad R = R^{a}_{a} \]

\[ R_{ab} = R_{ba} \rightarrow n(n+1)/2 \text{ independent components.} \]

**traceless part: Weyl tensor**

\[ C_{abcd} = R_{abcd} - \frac{2}{n-2} \left( g_{a[c} R_{d]b} - g_{b[c} R_{d]a} \right) + \frac{2}{(n-1)(n-2)} g_{a[c} g_{d]b} R \]

Same symmetries as the Riemann tensor + tracelessness,
\( (n-3)n(n+1)(n+2)/12 \) independent components. Increases as \( n^4 \) with dimension!
Einstein’s equations

**Einstein’s equations**

<table>
<thead>
<tr>
<th>curvature</th>
<th>$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$</th>
<th>matter sources</th>
</tr>
</thead>
</table>

2nd order non-linear PDEs for unknown metric $g_{ab}$. We are mostly interested in the case $T_{ab} = 0$ ("vacuum"),

**Einstein spacetimes (vacuum spacetimes)**

$$R_{ab} = \frac{2}{n-2} \Lambda g_{ab}$$

Limiting case for almost flat metric and small velocities of the matter sources:

**Newtonian gravity**

<table>
<thead>
<tr>
<th>gravitational potential</th>
<th>$\Delta \phi = 4\pi G \rho$</th>
<th>matter density</th>
</tr>
</thead>
</table>

V. Pravda, Institute of Mathematics, AS CR, Prague
Universal, almost universal and related spacetimes
Einstein’s equations are too involved and thus one has to make some simplifying assumptions:

- **symmetries** - this is how the Schwarzschild black hole has been discovered in 1915.

- **algebraic type of the Weyl tensor** - Petrov-Penrose type - this is how the Kerr black hole has been discovered in 1963.

Penrose diagram - algebraic types of the Weyl tensor in 4d
Algebraic classification of tensors - null frame

On a \( d \) dimensional Lorentzian manifold we will work in the frame

\[
\mathbf{n}, \, \ell, \, \mathbf{m}^{(i)}, \quad i, j, k = 2 \ldots d - 1,
\]

with two null vectors \( \mathbf{n}, \ell \)

\[
\ell^a \ell_a = n^a n_a = 0, \quad \ell^a n_a = 1, \quad a = 0 \ldots d - 1
\]

and \( d - 2 \) spacelike vectors

\[
\mathbf{m}^{(i)}, \quad m^{(i)a} m^{(j)}_a = \delta_{ij}, \quad i, j, k = 2 \ldots d - 1.
\]

The metric has the form

\[
g_{ab} = 2 \ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b.
\]
Lorentz transformations

The group of ortochronous Lorentz transformations is generated by null rotations

\[ \hat{\ell} = \ell + z_i m(i) - \frac{1}{2} z^i z_i n, \quad \hat{n} = n, \quad \hat{m}(i) = m(i) - z_i n, \]

spins

\[ \hat{\ell} = \ell, \quad \hat{n} = n, \quad \hat{m}(i) = X^i_j m(j), \]

and boosts

\[ \hat{\ell} = \lambda \ell, \quad \hat{n} = \lambda^{-1} n, \quad \hat{m}(i) = m(i). \]

A quantity \( q \) has a boost weight \( b \) if it transforms under a boost according to

\[ \hat{q} = \lambda^b q. \]
Boost order

Boost order of a tensor $T$ is the maximum boost weight of its frame components.

Proposition

Let $\ell, n, m^{(i)}$ and $\hat{\ell}, \hat{n}, \hat{m}^{(i)}$ be two null-frames with $\ell$ and $\hat{\ell}$ scalar multiples of each other. Then, the boost order of a given tensor is the same relative to both frames.

Thus boost order of a tensor depends (only) on the choice of a null direction - $\ell$. 
Project the tensor $T$ on the null frame and sort its components by their boost weight

$$T = \sum_{b=b_{\text{min}}}^{b_{\text{max}}} (T)_{(b)}.$$

<table>
<thead>
<tr>
<th>algebraic type</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>type G</td>
<td>general</td>
</tr>
<tr>
<td>type I</td>
<td>compts with $b = b_{\text{max}}$ can be set to zero</td>
</tr>
<tr>
<td>type II</td>
<td>compts with $b &gt; 0$ can be set to zero</td>
</tr>
<tr>
<td>type III</td>
<td>compts with $b \geq 0$ can be set to zero</td>
</tr>
<tr>
<td>type N</td>
<td>compts with $b &gt; b_{\text{min}}$ can be set to zero</td>
</tr>
</tbody>
</table>
Bivector

**Example:** for a bivector $K_{ab} = -K_{ba}$, we have $b_{\text{max}} = 1$,

\[
K_{ab} = 2K_{0i} \, n_{[a} m^i_{b]} + 2K_{01} \, n_{[a} \ell_{b]} + K_{ij} \, m^i_{[a} m^j_{b]} + 2K_{1i} \, \ell_{[a} m^i_{b]}.
\]

If $K_{0i} = 0$ then $\ell$ is aligned. If in addition, $K_{01} = K_{ij} = 0$, then the multiplicity of $\ell$ is 2.

In general types G, II, D, N (in even dimensions: type II or more special, in 4D: D, N)
\[ C_{abcd} = C\{abcd\} \equiv \frac{1}{2} (C_{[ab][cd]} + C_{[cd][ab]}), \quad C^c_{acb} = 0, \quad C_a[bc] = 0. \]

\[ C = (C)^{(2)} + (C)^{(1)} + (C)^{(0)} + (C)^{(-1)} + (C)^{(-2)}. \]

\[ C_{abcd} = \begin{aligned} &4C_{0i0j} n\{a m_b n_c m_d \} \quad \text{boost weight } 2 - \text{type G} \\
&+ 8C_{010j} n\{a l_b n_c m_d \} + 4C_{0ijk} n\{a m_b m_c m_d \} \\
&+ 4C_{0101} n\{a l_b m_c m_d \} + 4C_{01ij} n\{a l_b m_c m_d \} \\
&+ 8C_{0i1j} n\{a m_b l_c m_d \} + C_{ijkl} n\{a m_b m_c m_d \} \\
&- 1, \quad \text{III} \\
&+ 8C_{101j} l\{a n_b l_c m_d \} + 4C_{1ijk} l\{a m_b m_c m_d \} \\
&- 2, \quad \text{N} \\
&+ 4C_{1i1j} l\{a m_b l_c m_d \} . \end{aligned} \]
In particular, **for type N** the Weyl tensor in an appropriate null frame

\[ C_{abcd} = 4 C_{1i1j} \ell_a m^{(i)}_b \ell_c m^{(j)}_d, \quad \text{where} \quad T_{abcd} \equiv \frac{1}{2} (T_{[ab][cd]} + T_{[cd][ab]}). \]

Equivalently, for type N the null vector \( \ell \) obeys [Ortaggio, CQG 2009]

\[ C_{ab[cd}\ell_e] = 0. \]

Note that for \( d > 4 \) this is **not** equivalent to the standard 4D Bel-Debever condition \( C_{abcd} \ell^d = 0. \)
Bel-Debever criteria [Ortaggio 2009]

\[ \ell \text{ is a WAND (type I) } \iff \ell[eC_a]_{bc}[d\ell_f] \ell^b \ell^c = 0, \]

\[ \ell \text{ is a WAND of multiplicity } \geq 2 \text{ (type II) } \iff \ell[eC_a]_b[cd\ell_f] \ell^b = 0, \]

\[ \ell \text{ is a WAND of mult. } \geq 3 \text{ (type III) } \iff \ell[eC_ab]_{cd\ell_f} = 0 = C_{abc}[d\ell_e] \ell^c, \]

\[ \ell \text{ is a WAND of multiplicity } 4 \text{ (type N) } \iff C_{ab[cd\ell_e]} = 0. \]

It turns out that most of the known exact solutions of higher-dimensional Einstein gravity are also algebraically special. For example, Myers-Perry black holes are, similarly as Kerr of type D, etc.
Spin coefficients

Ricci rotation coefficients $L_{ab}$, $N_{ab}$ and $\dot{M}_{ab}$ are defined by

$$\ell_{a;b} = L_{cd} m_a^{(c)} m_b^{(d)} \ , \ n_{a;b} = N_{cd} m_a^{(c)} m_b^{(d)} \ , \ m_{a;b}^i = M_{cd} m_a^{(c)} m_b^{(d)} ,$$

where $a, b = 0 \ldots d - 1$.

For example

$$\ell_{a;b} = L_{11} \ell_a \ell_b + L_{10} \ell_a n_b + L_{1i} \ell_a m_b^{(i)} + L_{i1} m_a^{(i)} \ell_b + L_{i0} m_a^{(i)} n_b + L_{ij} m_a^{(i)} m_b^{(j)} .$$

$$\rho_{ij} \equiv L_{ij} \ - \ \text{optical matrix}$$
Let us decompose optical matrix $\rho_{ij}$ into its tracefree symmetric part $\sigma_{ij}$ (shear), its trace $\theta$ (expansion) and its antisymmetric part $A_{ij}$ (twist)

$$
\rho_{ij} = \sigma_{ij} + \theta \delta_{ij} + A_{ij},
$$

$$
\sigma_{ij} \equiv L(ij) - \frac{1}{d-2} L_{kk} \delta_{ij},
\theta \equiv \frac{1}{d-2} L_{kk},
A_{ij} \equiv L[ij].
$$

$\ell$ is geodetic iff $L_{i0} = 0$.

**Optical scalars:** other scalar quantities (apart from expansion) out of $\ell_{a;b}$: shear and twist

$$
\sigma^2 \equiv \sigma_{ii}^2 = \sigma_{ij} \sigma_{ji},
\omega^2 \equiv -A_{ii}^2 = -A_{ij} A_{ji}.
$$
If $\ell$ is \textit{affinely parametrized}, i.e. $L_{10} = 0$, the optical scalars take the form

$$\sigma^2 = \ell(a;b)\ell^{(a;b)} - \frac{1}{d-2} (\ell^a;_a)^2, \quad \theta = \frac{1}{d-2} \ell^a;_a, \quad \omega^2 = \ell[a;b]\ell^{a;b}.$$ 

\textbf{Definition}

Spacetimes admitting a null geodesic field $\ell$ with the optical matrix of the form

- $\rho_{ij} = 0$ \quad Kundt
- $\rho_{ij} \propto \delta_{ij}$ \quad Robinson-Trautmann
VSI spacetimes in higher dimensions

**Definition of VSI spacetimes**

VSI spacetimes are spacetimes with all curvature invariants constructed from the Riemann tensor and its derivatives being zero.

**VSI theorem [A. Coley, R. Milson, V. P., A. Pravdová, CQG 2004c]**

In a $d$-dimensional Lorentzian spacetime, all curvature invariants of all orders vanish if and only if there exists an aligned non-expanding, non-twisting, shearfree, geodetic null direction along which the Riemann tensor has negative boost order.

VSI $\iff$ Kundt spacetimes of Weyl type III and Ricci type III
Explicit form of VSI metrics

[Coley et al. 03,06]

VSI spacetimes admit a Kundt metric

\[ ds^2 = 2du \left[ dr + H(u, r, x)du + W_\alpha(u, r, x)dx^\alpha \right] + \delta_{\alpha\beta}dx^\alpha dx^\beta, \]

with

\[ W_\alpha(u, r, x) = -\delta_{\alpha, 2} \frac{2\epsilon}{x^2} r + W^{(0)}_\alpha(u, x), \]

\[ H(u, r, x) = \frac{\epsilon r^2}{2(x^2)^2} + rH^{(1)}(u, x) + H^{(0)}(u, x), \quad (\epsilon = 0, 1). \]

The value of \( \epsilon = 0, 1 \) specifies to which of the two main subclasses \( \tau = 0 \) and \( \tau \neq 0 \) of the Kundt family the spacetime belongs.
Einstein gravity

\[ \int d^n x \sqrt{-g} \frac{1}{\kappa} (R - 2\Lambda_0) \]

Quadratic gravity

\[ S = \int d^n x \sqrt{-g} \left( \frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 + \gamma \left( R_{abcd}^2 - 4 R_{ab}^2 + R^2 \right) \right) \]

\[ \Rightarrow \text{quadratic gravity field equations} \quad \text{[Gullu, Tekin, Phys. Rev. D, 2009]} \]

\[ \frac{1}{\kappa} \left( R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_0 g_{ab} \right) + 2\alpha R \left( R_{ab} - \frac{1}{4} R g_{ab} \right) + (2\alpha + \beta) \left( g_{ab} \nabla^c \nabla_c - \nabla_a \nabla_b \right) R \\
+ 2\gamma \left( R R_{ab} - 2 R_{acbd} R_{cd} + R_{acde} R_{b}^{\ cde} - 2 R_{ac} R_{b}^{\ c} - \frac{1}{4} g_{ab} \left( R^{2}_{cdef} - 4 R_{cd}^2 + R^2 \right) \right) \\
+ \beta \nabla^c \nabla_c \left( R_{ab} - \frac{1}{2} R g_{ab} \right) + 2\beta \left( R_{acbd} - \frac{1}{4} g_{ab} R_{cd} \right) R_{cd} = 0. \]
Obviously, it is very difficult to find exact vacuum solutions to quadratic gravity [Gullu, Gurses, Sisman, Tekin, Phys. Rev. D, 2011]. However, one can try to search for vacuum solutions of Einstein gravity, for which the **quadratic gravity terms are proportional to the metric**. Such spacetimes are “**immune**” to the quadratic gravity corrections and are **vacuum solutions to both** Einstein gravity and quadratic gravity.

This approach was taken in [T. Málek, V. P., Phys. Rev. D, 2011] and the result proven there, relevant to this talk, is

**Proposition**

In arbitrary dimension, all Weyl type N Einstein spacetimes are exact vacuum solutions of quadratic gravity.

Note that many such solutions are known.
Definition

A metric is called $k$-universal if all conserved symmetric rank-2 tensors constructed from the metric, the Riemann tensor and its covariant derivatives up to the $k^{\text{th}}$ order are multiples of the metric. If a metric is $k$-universal for all $k$ then it is called universal.

\[
T^{[ab]} = 0, \quad T^{ab}_{\ ;b} = 0 \Rightarrow T_{ab} = \lambda g_{ab}.
\]

Note that universal spacetimes are vacuum solutions to all theories with the Lagrangian of the form

\[
L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_p} R_{bcde})
\]

Relation to VSI spacetimes

Definition of VSI spacetimes

VSI spacetimes are spacetimes with all curvature invariants constructed from the Riemann tensor and its derivatives being zero.

All VSI spacetimes identified in [Coley, Milson, V.P., Pravdová, CQG, 2004]. Already there we anticipated that a “wide range” of VSIs will be universal.
Main results (arbitrary type and dimension)

Definition of CSI spacetimes

CSI spacetimes are spacetimes with all curvature invariants (constructed from the Riemann tensor and its derivatives) being constant.

Conjectured in [Coley, Hervik, 2011], proven in [Hervik, V. P., Pravdová, 2014]:

Proposition 1

A universal spacetime is necessarily a CSI spacetime.

\[ U \subseteq \text{CSI} \]

Note that many CSIs are not universal!
Main results for type N and III

Sufficient part discussed without a proof already in [Coley, Gibbons, Hervik, Pope 2008]:

**Proposition 2 - Necessary and sufficient conditions for Weyl type N universal spacetimes** [Hervik, V. P., Pravdová, 2014]

A type N spacetime is universal if and only if it is an Einstein Kundt spacetime.

**Proposition 3 - Sufficient conditions for Weyl type III universal spacetimes** [Kuchynka, Málek, V. P., Pravdová, 2019]

Weyl type III Einstein Kundt spacetimes obeying

\[ F_0 \equiv C^a_{cde} C^{bcde} = 0, \quad F_2 \equiv C^{pqrs}_{;a} C_{pqrs}_{;b} = 0 \]

are universal.
Explicit examples for type II (including D)

**Proposition 4 - type D universal spacetimes**

\[ M = M_0 \times M_1 \times \cdots \times M_{N-1} \quad (M_0 \text{ is Lorenzian}) \]

\[ M_\alpha, \alpha = 0 \ldots N - 1 \text{ are maximally symmetric spaces of dimension } n_\alpha \text{ and the Ricci scalar } R_\alpha \]

- \( M \) is Einstein \( \iff \frac{R_\alpha}{n_\alpha} = \frac{R_0}{n_0}, \forall \alpha \)
- \( M \) is universal \( \iff R_\alpha = R_0, n_\alpha = n_0, \forall \alpha \)

Thus in contrast with Einstein spacetimes, in this way, we can construct universal spacetimes only for **composite** number dimensions.

**Kundt extensions:**

**Proposition 5 - type II universal spacetimes**

When \( M_0 \) is type N or III universal, \( M \) is type II universal.
More general universal spacetimes (e.g. generalized Ghanam-Thompson) likely to exist (we have a proof of 2-universality), however, no example is known for prime number dimensions. In fact, we have proven:

**Proposition 6**

In five dimensions, type II (and D) universal spacetimes do not exist.
Proof of sufficiency for type N

Using higher-dimensional NP/GHP formalism one can prove

**Proposition**

For type N Einstein Kundt spacetimes, the boost order of $\nabla^{(k)} C$ with respect to the multiple WAND is at most $-2$.

Consequently, any rank-2 tensor quadratic (or of higher order) in $\nabla^{(k)} C$ vanishes. After proving a similar result for tensors linear in $\nabla^{(k)} C$, we arrive at

**Proposition**

For type N Einstein Kundt spacetimes, all rank-2 tensors constructed from the Weyl tensor and its derivatives vanish. These spacetimes are thus universal.
Proof of the necessary conditions for type N

Definition

CSI spacetimes are spacetimes with all curvature invariants being constant.

Proposition 1 (again)

A universal spacetime is necessarily a CSI spacetime.

One can show that for type N Einstein spacetimes, a curvature invariant

$$I_N \equiv C^{a_1 b_1 a_2 b_2; c_1 c_2} C_{a_1 d_1 a_2 d_2; c_1 c_2} C^{e_1 d_1 e_2 d_2; f_1 f_2} C_{e_1 b_1 e_2 b_2; f_1 f_2}$$

is constant iff the spacetime is Kundt.
Field equations \cite{Iyer:1994gr} \[ L = L(g_{ab}, R_{abcd}, \nabla_{a_{1}} R_{bcde}, \ldots, \nabla_{(a_{1}\ldots a_{p})} R_{bcde}) \]

\[ -T^{ab} = \frac{\partial L}{\partial g_{ab}} + E^{a}_{cde} R^{bcde}_{cde} + 2\nabla_{c} \nabla_{d} E^{acdb} + \frac{1}{2} g^{ab} L \quad \text{conserved} \]

\[ E^{bcde} = \frac{\partial L}{\partial R_{bcde}} - \nabla^{a_{1}} \frac{\partial L}{\partial \nabla_{a_{1}} R_{bcde}} + \cdots + (-1)^{p} \nabla_{(a_{1}\ldots a_{p})} \frac{\partial L}{\partial \nabla_{(a_{1}\ldots a_{p})} R_{bcde}} \]

trace \[ -T^{a}_{a} = g_{ab} \frac{\partial L}{\partial g_{ab}} + E_{bcde} R^{bcde}_{cde} + 2\nabla_{c} \nabla_{d} E^{cda}_{a} + \frac{D}{2} L \]

Let \( I \) be an arbitrary curvature invariant. Taking \( L = I, I^{2}, \ldots, I^{p} \)
with sufficiently high \( p \), one arrives at \[ I = \text{constant} \Rightarrow U \subset \text{CSI}. \]
Non existence of type II universal spacetimes in 5d

- Prove that $S_{ab}^{(2)} \equiv C_{acde}C_{b}^{cde}$ is conserved for $U$.
- Rewrite $S_{ab}^{(2)} = Kg_{ab}$ in terms of frame components

$$2\Phi\Phi_{ij}^{S} - 3\Phi_{ik}^{A}\Phi_{jk}^{A} - \Phi_{ik}^{S}\Phi_{jk}^{S} + \delta_{ij}(2\Phi_{kl}^{S}\Phi_{kl}^{S} - \Phi^2) = \frac{K}{4}\delta_{ij}.$$ 

- Prove that $S_{ab}^{(3)} = C^{cdef}C_{cdga}C_{efb}^{g}$ is also conserved for $U$.
- Rewrite $S_{ab}^{(3)} = K'g_{ab}$ as above.
- Show that the two above sets of equations imply $\Phi_{ij} = 0$ which gives type III.
A generalization of the Ghanam-Thompson metric

\[ ds^2 = 2du dv + (\lambda v^2 + H(u, x_\alpha, y_\alpha))du^2 + \frac{1}{|\lambda|} \sum_{\alpha=1}^{N-1} (dx_\alpha^2 + s^2(x_\alpha)dy_\alpha^2), \]

where \( s(x_\alpha) = \sin(x_\alpha) \) for \( \lambda > 0 \), \( s(x_\alpha) = \sinh(x_\alpha) \) for \( \lambda < 0 \).

The metric is Einstein iff \( H \) obeys

\[ \Box H = \left[ \sum_{\alpha=0}^{N-1} \Box^{(\alpha)} \right] H = 0, \quad \Box^{(\alpha)} H \equiv \nabla^{a(\alpha)} \nabla_{a(\alpha)}. \]

This is a type II Kundt metric for which for arbitrary \( k \) boost order of \( \nabla^{(k)} C \leq -2 \). Thus its derivatives behave like type N . . .
Proposition 7

Generalized GT metric obeying the Einstein condition

\[
\left[ \sum_{\alpha=0}^{N-1} \Box^{(\alpha)} \right] H = 0 \quad \text{is 0-universal. If in addition}
\]

\[
\left[ \sum_{\alpha=0}^{N-1} (\Box^{(\alpha)})^2 \right] H = 0.
\]

then it is 2-universal.

Conjecture 8

Generalized GT metric obeying the following set of \( N \) equations

\[
\left[ \sum_{\alpha=0}^{N-1} (\Box^{(\alpha)})^P \right] H = 0,
\]

where \( P = 1 \ldots N \), is universal.
Almost universal spacetimes

definition - TN spacetimes

TN spacetimes are spacetimes, for which all symmetric rank-2 tensors constructed polynomially from a metric, the Riemann tensor and its covariant derivatives of an arbitrary order have the form

\[ T_{ab} = \lambda g_{ab} + \phi \ell_a \ell_b \]

for a constant \( \lambda \) and a function \( \phi \) (that both depend on the particular tensor \( T_{ab} \)) and a fixed null vector \( \ell \).

TN(traceless type N) - all rank-2 tensors constructed from the Riemann tensor and its covariant derivatives of an arbitrary order are of traceless type N. Just one non-trivial field equation.
**Definition - TNS spacetimes**

TNS spacetimes are TN spacetimes, for which all symmetric rank-2 tensors constructed polynomially from a metric, the Riemann tensor and its covariant derivatives of an arbitrary order reduce to

$$T_{ab} = \lambda g_{ab} + \sum_{n=0}^{N} a_n \Box^n S_{ab},$$

where $S_{ab}$ is the traceless Ricci tensor and constants $\lambda$ and $a_i$ depend on the particular tensor $T_{ab}$. 
Proposition - Sufficient conditions for TN spacetimes

All $d$-dimensional Kundt spacetimes of Weyl type III and traceless Ricci type N or more special are TN.

Proposition - Sufficient conditions for Weyl type III TNS spacetimes

Kundt spacetimes of Weyl type III and traceless Ricci type N obeying

$$F_0 \equiv C^a_{cde} C^{bcde} = 0, \quad F_2 \equiv C^{pqrs}_{;a} C_{pqrs}_{;b} = 0$$

are TNS.
Universal spacetimes are vacuum solutions of all theories with

\[ L = L(g_{ab}, R_{abcd}, \nabla a_1 R_{bcde}, \ldots, \nabla a_1 \ldots a_p R_{bcde}) \]

\[ U \subset \text{CSI} \]

A type N spacetime is universal if and only if it is an Einstein Kundt spacetime.

A subclass of type III Kundt spacetimes is also universal.

These results can be used to construct explicit examples of universal metrics. They belong to the Kundt CSI class

\[ ds^2 = 2du [dr + H(u, r, x^\gamma)du + W_\alpha(u, r, x^\gamma)dx^\alpha] + g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta, \]

where \( H \) is quadratic in \( r \), \( W_\alpha \) are linear in \( r \), and \( g_{\alpha\beta}(x^\gamma) \) is a locally homogeneous metric.
Type D universal metrics can be constructed for all composite number dimensions.

The type N universal metrics can be used to construct type II universal metrics.

It seems that more general type II generalized GT metrics are also universal.

We have no examples of type II universal metrics for prime number dimensions.

We showed that in five dimensions, type II universal metrics do not exist.

D=7, 11, ... ??