On construction of differential $\mathbb{Z}$-graded varieties

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April 8, 2026

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NQ-Manifolds

Definition

A (splitted) NQ-manifold (E, Q) is a $\mathbb{N}$-graded manifold $E = \big(M, \Gamma(S(\oplus_{i\in\mathbb{N}} E^*_{-i}))\big)$ together with a degree $+1$ homological vector field $Q \in \mathfrak{X}_{+1}(E)$, i.e., which satisfies $Q^2 = \frac{1}{2}\,[Q, Q] = 0$.

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Algebraic point of view

We replace $C^\infty(M)$ by an arbitrary $\underline{\text{unital commutative algebra } \mathscr{A}}$ and $\Gamma(E_{-i})$ by a $\underline{\text{projective } \mathscr{A}\text{-module}}$.

Tangent complexes and Lie-Rinehart algebras

Every Lie ∞-algebroid comes equipped with a sequence:

$$\begin{tikzcd} \cdots \arrow[r] & E_{-3} \arrow[r, "\ell_1"] & E_{-2} \arrow[r, "\ell_1"] & E_{-1} \arrow[r, "\rho", red] & TM \end{tikzcd}$$

which is a complex of vector bundles on M.

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which is a complex of vector bundles on M.

1. The "red" map is a morphism of brackets.
2. The 2-bracket ℓ_2 does not satisfy Jacobi but

$$\left(\frac{\Gamma(E_{-1})}{\ell_1(\Gamma(E_{-2}))}, \ell_2, \rho \right)$$

   is a Lie-Rinehart algebra over $C^\infty(M)$.

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2. a ($\mathscr{A}$-linear) Lie algebra map $\rho\colon \mathscr{M} \longrightarrow \mathrm{Der}(\mathscr{A})$ satisfying

$$[u, fv]_{\mathscr{M}} = \rho(u)[f]v + f[u, v]_{\mathscr{M}} \ \ \text{(Leibniz)}$$

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Lie-Rinehart algebras are "nice" but with

1. No projective module!
2. No connections!
3. Bad differential geometry!

Results

Let $(\mathscr{M}, [\cdot\,,\cdot]_{\mathscr{M}}, \rho)$ be a Lie-Rinehart algebra over $\mathscr{A}$.

Theorem (PhD, J. Algebra, 2022)

Any resolution by free $\mathscr{A}$-modules of $\mathscr{M}$

$$\cdots \xrightarrow{\ell_1} \mathscr{E}_{-3} \xrightarrow{\ell_1} \mathscr{E}_{-2} \xrightarrow{\ell_1} \mathscr{E}_{-1} \xrightarrow{\pi} \mathscr{M} \longrightarrow 0$$

can be given a (unique up to homotopy) Lie ∞-algebroid structure $(\mathscr{E}_\bullet, \ell_\bullet, \rho)$ whose unary bracket is ℓ_1. *It is called the universal Lie ∞-algebroid of $\mathscr{M}$.*

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Losses	Many more structures for $\mathscr{M}$
Gains	Projectives modules, free modules, defined on generators

What the theorem says is

Acyclic Lie ∞-algebroids $\simeq$ Lie-Rinehart algebras

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Definition

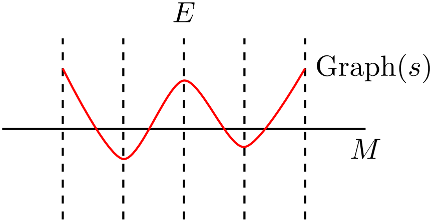
A submodule $\mathscr{F} \subseteq \mathfrak{X}(M)$ is said to be solvable if it admits a projective resolution of the form

$$\begin{tikzcd} \cdots \arrow[r] & \Gamma(E_{-3}) \arrow[r, "\mathrm{d}"] & \Gamma(E_{-2}) \arrow[r, "\mathrm{d}"] & \Gamma(E_{-1}) \arrow[r, "\rho"] & \mathscr{F} \arrow[r] & 0 \end{tikzcd}$$

My (PhD) result is valid even for non-solvable singular foliations!

Why $\mathbb{Z}$-graded extensions?

Let $E \to M$ be a vector bundle over a smooth manifold M and $s\colon M \to E$ a section, and let $\Sigma = \{s = 0\}$



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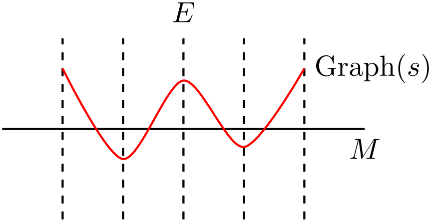

There exists a rather unusual Lie ∞-algebroid (negatively graded Q-manifold),

$$(\Gamma(\wedge^\bullet E[-1]^*), Q = \iota_s)$$

where ι_s is the contraction with s.

The 0-th homology is

$$H^0((\Gamma(\wedge^\bullet E[-1]^*), Q)) \simeq\ C^\infty(M)/\mathcal{I}_s$$

with $\mathcal{I}_s$ being the ideal generated by functions of the form $\langle \xi, s \rangle$ for $\xi \in \Gamma(E^*)$.

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- What is the universal Lie ∞-algebroid associated with $\mathcal{I}_s$ (the derivations that preserve $\mathcal{I}_s$)?

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- What is the universal Lie ∞-algebroid associated with $\mathcal{I}_s$ (the derivations that preserve $\mathcal{I}_s$)?
- How is it related to the negatively graded manifold associated with s?

$\mathbb{Z}$-graded extensions

One can always choose a negatively graded dg-variety so that there is no homology in degree less than or equal to -1, namely a Koszul-Tate resolution of $C^\infty(M)/\mathcal{I}_s$

$\mathbb{Z}$-graded extensions

Definition

Let $\mathscr{I}$ be an ideal of an algebra $\mathscr{A}$. The Koszul-Tate resolution of $\mathscr{A}/\mathscr{I}$ is a negatively graded Q-variety $(\mathscr{E}^-, \delta)$ over $\mathscr{A}$ such that

- $\mathscr{E}^- \simeq S(\oplus_{i\leq -1}\mathscr{V}_i)$, for some collection of projective $\mathscr{A}$-modules $\mathscr{V}_i$.
- the homology, $H^{-i}(\mathscr{E}^-, \delta) = 0$ for $i \geq 1$ and $H^0(\mathscr{E}^-, \delta) = \mathscr{A}/\mathscr{I}$.

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There exists an explicit construction of KT given by trees!

Arborescent Koszul-Tate resolution

Ingredients

- A projective resolution of $\mathscr{A}/\mathscr{I}$

$$\mathfrak{M}\colon\ \cdots \xrightarrow{d} \mathfrak{M}_{-i} \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{M}_{-1} \xrightarrow{d} \mathscr{A} \longrightarrow 0$$

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- The $\mathscr{A}$-module of symmetric decorated trees $\mathcal{T}ree[\mathfrak{M}]$, together with a map

$$\psi\colon \mathcal{T}ree[\mathfrak{M}] \to \mathfrak{M}$$

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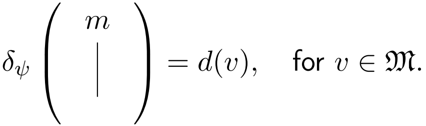
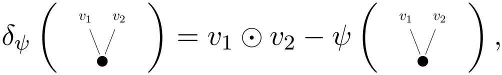
- A derivation $\delta_\psi : S(\mathcal{T}ree(\mathfrak{M})) \longrightarrow S(\mathcal{T}ree(\mathfrak{M}))$ of degree $+1$ that is defined recursively.

Arborescent Koszul-Tate resolution

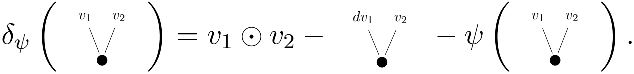
For example



for $v \in \mathfrak{M}$.



for $v_1, v_2 \in \mathfrak{M}_{-1}$.



if $|v_1| < -1, |v_2| = -1$.

Arborescent Koszul-Tate resolution

Theorem (Hancharuk, 2023)

There exists a map $\psi\colon \mathcal{T}ree[\mathfrak{M}] \to \mathfrak{M}$ *of degree* $+1$ *such that* $(S(\mathcal{T}ree[\mathfrak{M}]), \delta_\psi)$ *is a Koszul-Tate resolution of* $\mathscr{A}/\mathscr{I}$*.*

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$$\begin{tikzcd}
(S(\mathcal{T}ree[\mathfrak{M}]), \delta_\psi) \arrow[loop left] \arrow[r, bend left] & (\mathfrak{M}, d) \arrow[l, bend left]
\end{tikzcd}$$

Arborescent extension of $\mathbb{N}$-graded Q-varieties

It is natural to ask

Question 1

Is there a $\mathbb{Z}$-graded Q-variety $(\mathscr{E} = \hat{S}(\oplus \mathscr{V}_{i\in\mathbb{Z}}), Q)$ encoding both

- the Koszul–Tate resolution of $\mathscr{A} / \mathscr{I}$ (a singular space),

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Answer:

YES!

A. Kotov, C. Laurent-Gengoux, and V. Salnikov, J. of Geom. and Phys (2023).

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Question 2

To what extent the $\mathbb{Z}$-graded Q-variety $(\mathscr{E} = \hat{S}(\oplus \mathscr{V}_{i\in\mathbb{Z}}), Q)$ is computable? Is there an algorithm that terminates in a finite number of steps?

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Answer:

YES! (H-L, 2025, Arxiv)

If we take an arborescent Koszul-Tate resolution of $\mathscr{A}/\mathscr{I}$

Arborescent extension from $\mathscr{A}/\mathscr{I}$

Assume that the (smooth) Kähler module $\Omega_{\mathscr{A}/\mathbb{K}}$ is a projective $\mathscr{A}$-module. Let $\mathscr{I} \subset \mathscr{A}$ be a proper ideal of $\mathscr{A}$.

Theorem 1 (H-L, 2025, Arxiv)

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Then there is a $\mathbb{Z}$-graded extension $(\mathscr{E}, Q)$ such that

- **the homological vector field Q decomposes as**

$$Q = \delta_\psi + \nabla_\alpha + L_\beta$$

Arborescent extension from $\mathscr{A}/\mathscr{I}$

- the negative degree -1 component $Q_{(-1)}$ is δ_ψ.
-

$$Q_{(0)}(a) = \begin{cases} (\nabla_\alpha)_{(0)}(a) = \hat{Q}^+(a), \ \text{if } a \in \hat{\mathcal{A}}^+; \\ \\ (\nabla_\alpha)_{(0)}(a) - \underbrace{\mathfrak{r} \circ \mathrm{p}^{\geq 2} \circ \left(\left[\delta_\psi, (\nabla_\alpha)_{(0)}\right] + (L_\beta)_{(0)} \circ \delta_\psi \right)(a) - \beta_{(0)}(a)}_{(L_\beta)_{(0)}}, \ \text{if } a \in \mathcal{T}ree[\mathfrak{M}]_{(\bullet)}. \end{cases} \tag{12}$$

- For $N \geq 0$, the negative degree $N+1$ component $Q_{(N+1)}$ is given recursively as:

$$Q_{(N+1)}(a) = \begin{cases} (\nabla_\alpha)_{(N+1)}(a) = -\sum\limits_{\substack{i+j=N \\ 0 \leq i \leq j}} \mathfrak{r} \circ \mathrm{p}^{\geq 2} \circ \left[Q_{(i)}, Q_{(j)}\right](a) - \alpha_{(N+1)}(a), \ \text{if } a \in \mathcal{O} \\ \\ (\nabla_\alpha)_{(N+1)}(a) - \underbrace{\mathfrak{r} \circ \mathrm{p}^{\geq 2} \circ \left(\left[\delta_\psi, (\nabla_\alpha)_{(N+1)}\right] + (L_\beta)_{(N+1)} \circ \delta_\psi + \sum\limits_{\substack{i+j=N \\ 0 \leq i \leq j}} \left[Q_{(i)}, Q_{(j)}\right] \right)(a) - \beta_{(N+1)}(a)}_{(L_\beta)_{(N+1)}}, \\ \\ \text{if } a \in \mathcal{T}ree[\mathfrak{M}]_{(\bullet)} \oplus \mathcal{V}_{\geq 1}. \end{cases} \tag{13}$$

Arborescent extension from $\mathscr{A}/\mathscr{I}$

α and β are respectively $\mathbb{K}$-linear and $\mathscr{A}$-linear maps of degree $+1$

$$\begin{cases} \alpha\colon \mathscr{A} \to \mathfrak{M}\hat{\odot}\tilde{\mathscr{E}}^{+} \quad \text{with} \quad \alpha_{(0)} = 0 \\ \\ \beta\colon \mathcal{T}ree[\mathfrak{M}] \oplus \mathscr{V}_{\geq 1} \to \mathfrak{M}\hat{\odot}\tilde{\mathscr{E}}^{+} \quad \text{with} \quad \beta_{(0)}|_{\mathscr{V}_{\geq 1}} = 0 \end{cases} \tag{1}$$

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- the homological computations are restricted to the collection of $\mathscr{A}$-modules

$$\Big(\mathcal{T}ree[\mathfrak{M}]_{(i)},\ \mathfrak{M}_{(i+j-1)}\hat{\odot}\tilde{\mathscr{E}}_{j}^{+}\Big) \cup \Big(\mathscr{V}_{i},\ \mathfrak{M}_{(j)}\hat{\odot}\tilde{\mathscr{E}}_{i+j+1}^{+}\Big)$$

and

$$(\Omega_{\mathscr{A}/\mathbb{K}},\ \mathfrak{M}_{(i)}\hat{\odot}\tilde{\mathscr{E}}_{i+1}^{+}),\ i, j \geq 1.$$

Arborescent extension from $\mathscr{A}$

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We can drop the assumption on $\Omega_{\mathscr{A}/\mathbb{K}}$

Theorem 2

There is a $\underline{\text{very}}$ explicit formula!

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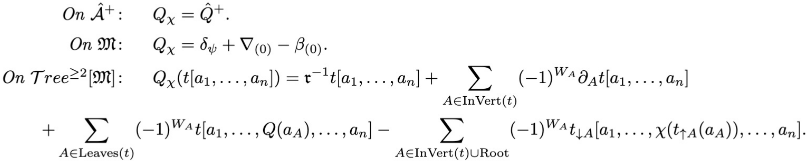

$$\begin{aligned}
\textit{On } \hat{\mathcal{A}}^+: \quad & Q_\chi = \hat{Q}^+. \\
\textit{On } \mathfrak{M}: \quad & Q_\chi = \delta_\psi + \nabla_{(0)} - \beta_{(0)}. \\
\textit{On } \mathcal{T}\!\mathit{ree}^{\geq 2}[\mathfrak{M}]: \quad & Q_\chi(t[a_1,\dots,a_n]) = \mathfrak{r}^{-1}t[a_1,\dots,a_n] + \sum_{A\in \mathrm{InVert}(t)} (-1)^{W_A}\partial_A t[a_1,\dots,a_n] \\
& + \sum_{A\in \mathrm{Leaves}(t)} (-1)^{W_A} t[a_1,\dots,Q(a_A),\dots,a_n] - \sum_{A\in \mathrm{InVert}(t)\cup \mathrm{Root}} (-1)^{W_A} t_{\downarrow A}[a_1,\dots,\chi(t_{\uparrow A}(a_A)),\dots,a_n].
\end{aligned}$$

Example

Let $\mathscr{A} = \mathbb{K}[x, y]$ and let $\mathscr{I} = \langle x^2, xy, y^2 \rangle$. Let $\mathscr{M} \subset \mathrm{Der}(\mathscr{A})$ be the Lie-Rinehart algebra made of derivations X of $\mathscr{A}$ that preserve $\mathscr{I}$, i.e., $X[\mathscr{I}] \subset \mathscr{I}$. The Lie-Rinehart algebra $\mathscr{M}$ is spanned by

$$\langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\rangle.$$

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$\mathscr{M}$ has a resolution by free $\mathscr{A}$-module:

$$0 \longrightarrow P_{-2} \xrightarrow{\ell} P_{-1} \xrightarrow{\rho} \mathscr{M} \tag{2}$$

with $P_{-1} \simeq \mathscr{A}^4$, $P_{-2} \simeq \mathscr{A}^2$.

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with $P_{-1} \simeq \mathscr{A}^4$, $P_{-2} \simeq \mathscr{A}^2$. The anchor map ρ:

$$\rho(e_1) = x\frac{\partial}{\partial x}, \quad \rho(e_2) = y\frac{\partial}{\partial x}, \quad \rho(e_3) = x\frac{\partial}{\partial y}, \quad \rho(e_4) = y\frac{\partial}{\partial y}.$$

The map ℓ :

$$\ell(u) = xe_2 - ye_1, \quad \ell(v) = xe_4 - ye_3.$$

where u, v are basis elements of P_{-2}.

The positive graded part $(S(\mathcal{V}_1 \oplus \mathcal{V}_2), \hat{Q}^+)$

Write ξ^1, ξ^2, ξ^3 for the basis of $\mathcal{V}_1 := P^*_{-1}$ and η^1, η^2 for the basis of $\mathcal{V}_2 := P^*_{-2}$.

$$\begin{aligned}
\hat{Q}^+ = \; & \xi^1 x \frac{\partial}{\partial x} + \xi^2 y \frac{\partial}{\partial x} + \xi^3 x \frac{\partial}{\partial y} + \xi^4 y \frac{\partial}{\partial y} + \eta^1 (x \frac{\partial}{\partial \xi^2} - y \frac{\partial}{\partial \xi^1}) \\
& + \eta^2 (x \frac{\partial}{\partial \xi^4} - y \frac{\partial}{\partial \xi^3}) + (\xi^2 \xi^3) \frac{\partial}{\partial \xi^1} + (\xi^1 \xi^2 + \xi^2 \xi^4) \frac{\partial}{\partial \xi^2} - (\xi^1 \xi^3 + \xi^3 \xi^4) \frac{\partial}{\partial \xi^3} \\
& - (\xi^2 \xi^3) \frac{\partial}{\partial \xi^4} + (\xi^2 \eta^2 - \xi^4 \eta^1) \frac{\partial}{\partial \eta^1} + (-\xi^1 \eta^2 + \xi^3 \eta^1) \frac{\partial}{\partial \eta^2}.
\end{aligned}$$

The negative graded part δ_ψ

It is obtained from a free resolution $(\mathfrak{M}, d)$ of $\mathscr{A}/\mathscr{I}$

$$0 \longrightarrow \mathfrak{M}_{-2} \xrightarrow{d} \mathfrak{M}_{-1} \xrightarrow{d} \mathscr{A} \longrightarrow 0$$

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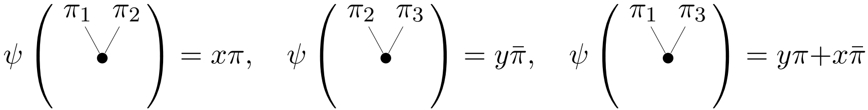
d is defined on the basis $\{\pi_1, \pi_2, \pi_3\}$ of $\mathfrak{M}_{-1}$ and $\{\pi, \bar{\pi}\}$ of $\mathfrak{M}_{-2}$ as

$$d(\pi_1) = x^2, \quad d(\pi_2) = xy, \quad d(\pi_3) = y^2,$$

$$d(\pi) = x\pi_2 - y\pi_1, \quad d(\bar{\pi}) = x\pi_3 - y\pi_2.$$

The negative graded part δ_ψ

The arborescent Koszul-Tate resolution $(S(\mathcal{T}ree[\mathfrak{M}]),\, \delta_\psi)$ is determined by:

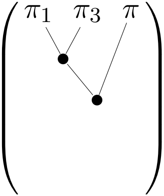


$$\psi\left(\begin{matrix}\pi_1 & & \pi_2\\ & \diagdown\,\diagup & \\ & \bullet & \\ & & \end{matrix}\right) = x\pi, \quad \psi\left(\begin{matrix}\pi_2 & & \pi_3\\ & \diagdown\,\diagup & \\ & \bullet & \\ & & \end{matrix}\right) = y\bar{\pi}, \quad \psi\left(\begin{matrix}\pi_1 & & \pi_3\\ & \diagdown\,\diagup & \\ & \bullet & \\ & & \end{matrix}\right) = y\pi + x\bar{\pi}$$

and 0 in all other cases.

The total $\mathbb{Z}$-graded description

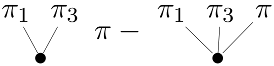
For example



$$Q\left(\quad \right) = (A) + (B) + (C) + (D) + (E).$$

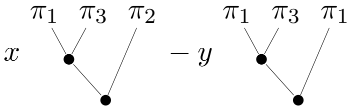
where

(A) is of the form



The total $\mathbb{Z}$-graded description

(B) is the contribution from d acting by derivation on leaves of the tree:



Note that terms where d acts on leaves π_1 or π_3 are zero.

Thank you for your attention!

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