Much works for general $G$, focus on $S_n$ (symmetric group)

Motivating problem: $\{1, \ldots, n\} \subseteq S_n$

$$A = \{ A_\sigma \}_{\sigma \in S_n} \quad n! \cdot x_n!$$

$$A_\sigma := \chi \cdot \#C(\sigma^{-1})$$

$x$: undetermined variable

$$\#C(\sigma) \quad \text{number of cycles of permutation } \sigma \in S_n$$

$$\det(x^2, x) = (x-1)x^2(x+1)$$

$$\det A = \prod_{k=0}^{n-1} (x^2-k^2) \chi_k$$

$\chi_k \in \mathbb{N}$ computed by $V_{\lambda}, V_{\lambda^*}$

A is the image of $\sum_{\sigma \in S_n} \chi \cdot \#C(\sigma) \in C[x] \otimes C[S_n]$ in $End(C[x] \otimes C[S_n])$.

where $\omega$ has factors $(X \pm X_i)$, $i = 1, \ldots, n$

$$X_k := \sum_{j \neq k} (jk) \quad \text{Jucys-Murphy elements in } C[S_n]$$

$sym$ funs in $X_i$ do generate center $Z(C[S_n])$

$$C[S_n] = \bigoplus_{\lambda = \emptyset}^{\dim(V_\lambda)} V_\lambda$$

$$\Rightarrow \det A = \prod_{k=0}^{n-1} (x^2-k^2) \chi_k$$

Preliminary exercise: $X_i, X_j \in C[S_n], 1 < i < j \leq n$

$$X_i = (1i) + (2i) + \cdots + (i-1i)$$

commute each other!

**Pf:** All contributions to $X_i X_j$ $X_j X_i$ are $(k,i)(m,j)$ $(m,j)(k,i)$

are equal if $k, \{i,j\}$ has exactly 4 elements (mut. diff.)
- remaining cases left are $k = m < i < j$ and $k < i = m < j$.

$$X_i X_j = X_j X_i$$

is equivalent to

$$\sum_{k=1}^{i-1} (k_i)(k_j) + \sum_{k=1}^{i-1} (k_i)(ij) = \sum_{k=1}^{i-1} (k_j)(k_i) + \sum_{k=1}^{i-1} (ij)(k_i)$$

But $(k_i)(k_j) = (k_j k_i) = (ij)(k_i)$

$(k_j)(k_i) = (k_i k_j) = (k_i)(ij) = (k_i)(k_j)$

$\Rightarrow$ both sides are

$$\sum_{k=1}^{i-1} (k_i) + (k_j)$$

hence equal.

The complexity of $S_n$ (compared to $GL(n)$) given by Coxeter relations $s_i s_{i+1} s_i = s_i$ (not the commut. rel.)

no Cartan subgroup. However,

$$\text{Mor}_{k-\text{alg.}} (k[G], \text{End}(V)) \cong \text{Mor}_{G_0} (G, GL(V))$$

so work with $C[S_n]$.

Young diagrams/tableaux (indirect approach) aux, const. needed.

Therefor,

(a) (Gelfand pair property) $S_{n+1}$ is simple (= multi. free) $\Rightarrow$ decomp. of $C[S_n]$-mod $V$ into irreps for $S_{n+1}$ is canonical, iterate... $S_1$-irrep. are 1-dim

(b) $Z_n \subset C[S_n]$ center, Gelfand-Tsetlin $GZ_n$

generated by $Z_1, \ldots Z_n$ acts diagonally in the $GZ$-basis for $V$ irreps $j$ maximal commutative

$$\dim = \sum \dim V_\lambda$$

in equiv.

irrep. $S_n$ (=> all vectors in $GZ_2$-basis is deter. by eigenvalues of generators $GZ$-algebra

$GZ$-algebra $\leftrightarrow$ Cartan subalgebra
(c) \( x_i = (1^i) + (2^i) + \ldots + (i-1, i) \in \mathbb{C}[S_n] \)

Young - Jacobi - Murphy (YJM) elements

generate GZ-algebra; to a GZ-vector we associate
\( x(v) = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \) \( a_i \in \text{Spec}(X_i) \) for \( i \)

\( \text{Spec}(n) := \{ x(v) \mid v \text{ is a GZ-vector} \} \)

By (i), GZ-vectors \( u, v \) \( u = v \iff x(u) = x(v) \)

\( \Rightarrow \# \text{Spec}(n) = \text{sum of dim } \mathcal{F}_\lambda, \lambda \vdash n \), \( \forall x \text{ equiv class of } S_n - \text{rep} \)

(d) The construction of \( \text{injection } \text{Spec}(n) \subseteq \text{Tab}(n) \)

"triples in Spec(n) whose GZ-vectors belong to the same irrep" → "standard Young tableau of the same shape"

Based on the analysis of commutation relations \( i = 1, \ldots, n \)

\[ X_i X_{i+1} = X_{i+1} X_i, \quad s_i^2 = 1, \quad s_i X_i + 1 = X_{i+1} s_i \]

\[ \mathbb{C}[X_i, X_{i+1}] \times \mathbb{C}[S_n], \quad H(2) \]

\( H(2) \leftrightarrow \text{gl}(2) \)
Some details related to particular points:

(2) $G$-T bases / $G$-T algebras for inductive chain of finite groups (simple reduction)

$$G_1 \leq G_2 \leq \ldots \leq G_n \leq \ldots$$

$G_n$ ... equivalence classes of finitely many $V_k$

Bratelli diagram: vertices ... elements of $\bigcup_{n \geq 1} \hat{G}_n$

edges ... $\lambda, \mu$ joined by $k$ (directed)

edges $(\lambda, \mu)$ if $\mu \in \hat{G}_n$ for some $n$

fulfil by

$$s_1 \leq s_2 \leq \ldots \leq s_n \leq \ldots$$

$t_{ij}$

$s_{i,j} \rightarrow s_{i,j}$ preserves $t_{ij}$

$V_k \rightarrow \bigoplus V_{\mu}$

all $\mu$ canonical; intertwining $V_k \Rightarrow \oplus V_{\mu}$

$$T = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n$$

$\lambda_i = \lambda_i$ if $\lambda_i \in \hat{G}_i$

The choice of $V_T \neq 0$ in $V_k \Rightarrow$ bases $\{ V_{\mu} \}$ of $V_k$

$GZ$-basis

(then $V_{k_i} = C[G_l] \cdot V_T$

Which elements of $C[G_l]$ act diagonally in this basis?

$$C[G_l] \rightarrow \bigoplus \text{End}(V_k)$$

$\lambda \in \hat{G}_l$

$g \rightarrow (V_k \mapsto V_{\lambda} \mid \lambda \in \hat{G}_n)$

$D(V_k) \ldots$ diagonal in the $GZ$-basis

$$\bigoplus_{\lambda \in G_l} D(V_k) \subseteq G[End(V_k)]$$
\[ Z_n := Z(C[G_n]) \quad \text{in} \quad C[G_n] \quad \text{Gelfand-Tsetlin algebra} \quad (G-Z) \]

\[ \text{commutative algebra} \quad G Z_n \text{ image of } \bigoplus_{\lambda \in \mathbb{N}_n} D(V_\lambda) \]

(\# elements in \( C[G_n] \) that act diagonally in the \( G Z_n \)-basis in \# irrep of \( G_n \))

\[ \text{Thus } G Z_n \text{ is max comm, } \dim = \sum_{\lambda \in \mathbb{N}_n} \dim(V_\lambda) \]

(Consequence: \( u, v \in G Z_n \) are same eigenvec \( \forall G Z_n \), then \( u = v \).)

\[(a) \quad \text{"Criterion" s} \]

\[ M \quad \text{a fin dim semi-simple } C\text{-alg,} \]

\[ \text{NSH} \quad \text{centralizer of } N \text{ in } M \]

Then: \( Z(M/N) \text{ is semi-simple} \)

- TFAE:
  - \( Z(M/N) \text{ is semi-simple} \)
  - The restriction of a fin dim \( C\text{-alg} \) irrep of \( M \to N \) is mult. free

\[ (A, \ast) \quad \text{an involutive alg.,} \]

\[ x x^\ast = x^\ast x \text{ normal} \]

\[ X - x^\ast \text{ self-adjoint} \]

Then \( A \text{ is commutative iff } \forall x \in A \text{ is normal} \)

\[ \text{(eg: } G, \mathbb{R}, C \text{) } \}

\[ c[G] \text{ is } K \dashv (\sum_{\lambda} c_{\lambda} g_{\lambda} = (\sum_{\lambda} c_{\lambda} g_{\lambda})^\ast \}

\[ c[G] \text{ is } K \dashv \text{complex of } \mathbb{R}[G] \]

The centralizer \( Z(C[S_n], C[S_{n-1}]) \text{ is commutative.} \)
\[ X_i = (i1) + (2i) + \cdots + (i-1, i) \in C[S_n] \]

**Young-Jucys-Murphy elements (YJM)**

\[ X_i = \sum_{\sigma \in S_i} \text{tr}(\sigma) \quad \text{and} \quad X_i = \sum_{\sigma \in S_{i-1}} \text{tr}(\sigma) \]

\[ Z_i \quad \Rightarrow \quad X_i \in G_n \quad \forall i \leq n \]

**Theorem:**

\[ Z_n \subseteq C[S_n], \quad C[S_{n-1}] \rightarrow C[S_n] \] \[ \bigcup_{Z_{n-1}} \]

\[ X_n = \frac{1}{n!} \sum_{i=1}^{n-1} (i, i+1) = \frac{1}{n!} \sum_{i<k} \frac{1}{k!} - \frac{1}{i!} \]

\[ <Z_{n-1}, X_n> \subseteq Z_{n-1} \]

\[ X_n^2 = \ldots \]

**Consequence:** The \( GZ \)-algebra is generated by YJM elements,

\[ GZ_n = <X_1, X_2, \ldots, X_n> \]

\[ GZ_n = <Z_1, \ldots, Z_n> \quad \text{and} \quad GZ_{n-1} = <X_1, \ldots, X_{n-1}> \Rightarrow GZ_n. \quad <GZ_{n-1}, X_n> \]

\[ Z_n \subseteq <Z_{n-1}, X_n> \subseteq <GZ_{n-1}, X_n> \]

**Remark:**

YJM elements are not in the center\( X_i \in Z_n \), \( k = 1, \ldots, n \).

The centralizer \( Z_{n-1, 1} := Z\left(C[S_n], C[S_{n-1}]\right) \) is

\[ Z_{n-1, 1} = <Z_{n-1}, X_n> \].
The branching of the chain \( C[S_2] \subset \cdots \subset C[S_n] \)
\((\leq \mathbb{Z}_{n-1}, 1 \text{ is commutative}) \) because \( \mathbb{Z}_{n-1}, 1 \leq \langle \mathbb{Z}_{n-1}, X_0 \rangle \).

The appearance of YJM elements: \( \exists! \ S_{n+1} \to S_n \)
commuting with \( L \& R \) \( S_n \) action,
"removes" \( n+1 \) from the cyclic notation of perm. in \( S_{n+1} \)
extend to \( C[S_{n+1}] \to C[S_n] \), then
the affine subset := intersection of pre-image
of the identity element of \( C[S_n] \) with
the centralizer of \( C[S_n] \) is spanned
by identity element and \( \frac{1}{n} X_0 \).