# Sasakian structures on Smale-Barden manifolds

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# Plan

Aims:

1. A (reasonably general) theory of Kähler (symplectic) orbifolds and Seifert bundles,
2. a description of (some) existence problems in Sasakian geometry,
3. an application of (1) to (2), a sample of results,
4. (very rough) description of methods of proofs.

# Contact forms

$$(M^{2n+1}, \eta), \eta \in \Omega^1(M), \eta \wedge (d\eta)^n \neq 0. \text{ everywhere on } M.$$

There exists $\xi \in \chi(M)$ such that

$$\eta(\xi) = 1, \xi \lrcorner d\eta = 0$$

(Reeb vector field).

$\xi$ determines a 1-dimensional foliation $\mathcal{F}_\xi$ (characteristic foliation) and the decomposition

$$TM = \mathcal{D} \oplus L_\xi, \mathcal{D} = \mathrm{Ker}\, \eta.$$

# Examples

- $M = \mathbb{R}^{2n+1}$, coordinates $(x_1, ..., x_n, y_1, ..., y_n, z), \eta = dz - \sum_{i=1}^n y_i dx_i, \eta \wedge (d\eta)^n \neq 0$.
- $M = S^{2n+1} \subset \mathbb{R}^{2n+2}$ with coordinates $(x_0, ..., x_n, y_0, ..., y_n)$ and

$$\eta = \sum_{i=0}^{n} x_i dy_i - y_i dx_i|_{S^{2n+1}}.$$

Check up:

$$\eta \wedge (d\eta)^n = 2^n n! (\sum x_i dx_0 \wedge dy_0 \wedge \cdots \hat{dx_i} \wedge dy_i \wedge \cdots \wedge dx_n \wedge dy_n -$$

$$\sum y_i dx_0 \wedge dy_0 \wedge \cdots \wedge dx_i \wedge \hat{dy_i} \wedge \cdots \wedge dx_n \wedge dy_n|_{S^{2n+1}}.$$

Take $\beta = \sum_{i=0}^n (x_i dx_i + y_i dy_i$ and check that

$$\beta \wedge \eta \wedge (d\eta)^n = 2^n n! \sum_{i=0}^{n} ((x_i)^2 + (y_i)^2) dx_0 \wedge dy_0 \wedge \cdots \wedge dx_n \wedge dy_n.$$

# Sasakian manifolds

We say that $(M, \eta)$ admits a Sasakian structure $(M, g, \xi, \eta, J)$, if

- there exists an endomorphism $J : TM \to TM$ such that $J^2 = -\operatorname{id} + \xi \otimes \eta$,
- $d\eta(JX, JY) = d\eta(X, Y)$, for all $X$, $Y$, and $d\eta(JX, X) > 0, X \in \operatorname{Ker} \eta$,
- $\xi$ is Killing ($\mathcal{L}_\xi g = 0$) with respect to the Riemannian metric

$$g(X, Y) = d\eta(JX, Y) + \eta(X)\eta(Y)$$

- the almost complex structure $I$ on the contact cone $C(M) = (M \times \mathbb{R}_+, t^2 g + dt^2)$ defined by

$$I(X) = J(X), X \in \operatorname{Ker} \eta, I(\xi) = t\frac{\partial}{\partial t}, I(t\frac{\partial}{\partial t}) = -\xi$$

  is integrable.

# Thinking about Sasakian manifolds

## Boothby-Wang bundle

- $(X, \omega)$ symplectic, $[\omega] \in H^2(X, \mathbb{Z}) \implies$
- $S^1 \to M \to X$ - principal circle bundle determined by $[\omega]$

## Theorem

The Boothby-Wang circle bundle admits a K-contact structure. If $(X, \omega)$ is Kähler, then $M$ is Sasakian.

## Example

Hopf bundle

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n,$$

$$S^{2n+1} \subset \mathbb{C}^{n+1},$$

$S^1$ acts on $S^{2n+1}$ by multiplication.

# Explanation of Boothby-Wang example

- Kobayashi's theorem: there exists a connection form $\eta \in \Omega^1(M, L(S^1)) = \Omega^1(M, \mathbb{R})$ such that $p^*\omega = d\eta$, $\implies$
- $\eta$ is a contact form, since $d\eta$ is non-degenerate on the horizontal distribution ($TM = \langle\xi\rangle \oplus \mathcal{H}$),
- using $\langle\xi\rangle \oplus \mathcal{H}$ define a Riemannian metric $g$ on $M$ in a way to get a Riemannian submersion with totally geodesic fibers $\implies \mathcal{L}_\xi g = 0$,
- if $J$ is integrable on $\mathcal{H}$ (or, $X$ is Kähler), then $I$ is integrable on $C(M)$ (non-trivial calculation of the tensor

$$N_I(U, V) = [IU, IV] - [U, V] - I[U, IV] - I[IU, V]$$

separately for vector fields in $\mathcal{H}$,and $\xi, t\frac{\partial}{\partial t}$).

# Reference

S. Kobayashi, *Topology of positively pinched Kähler manifolds*, Toholu Math. J. 15(1963), 121-139.

# General Sasakian manifolds

## Rukimbira's theorem

*If a manifold $M$ admits a Sasakian structure, it also admits a quasi-regular Sasakian structure.*

Quasi-regular $\iff$ Seifert bundle over Kähler orbifold determined by orbifold Kähler class.

## Aim:

Create a reasonably general theory of Seifert bundles over Kähler orbifolds and use it to solve (some) existence problems in Sasakian geometry.

# Advertising Sasakian geometry

## Some reasons (chosen at random)

- The structure $(M, g, \xi, \eta, J)$ is a tool to construct $\eta$-Einstein metrics $\mathrm{Ric}(g) = \lambda g + \nu\eta \otimes \eta, \lambda, \nu \in \mathbb{R}$.
- Quasi-regular Sasakian structures is a tool to construct Einstein metrics (Sasaki-Einstein metrics)
- Every simply connected Sasaki-Einstein manifold admits a Killing spinor (Friedrich-Kath theorem)

These have applications in mathematical physics, esp. in General Relativity, supergravity theories, supersymmetry etc....

On the other hand, there are obstructions to the existence of Sasakian structures (for example, Kollár's obstructions in dimension 5) $\Longrightarrow$

Existence problems for Sasakian structures yield interesting mathematics.

# Orbifolds: definition

- $X$, Hausdorff, paracompact with atlas $\mathcal{U} = \{(\tilde{U}_i, \varphi_i, \Gamma_i)\}$, $\tilde{U}_i$ open in $\mathbb{R}^c(\mathbb{C}^n)$, $\Gamma_i < GL(n, \mathbb{R})$ finite,
- $U_i = \tilde{U}_i/\Gamma_i$, $\varphi_i(\gamma p) = \varphi_i(p), p \in \tilde{U}_i$,
- $X = \cup_i \varphi_i(\tilde{U}_i) = \cup_i U_i$, $U_i$ open in $X$,
- $(\tilde{U}_i, \Gamma_i, \varphi_i)$ and $(\tilde{U}_j, \Gamma_j, \varphi_j)$, $U_i, U_j$ and $x \in U_j \cap U_j$ there exists

  $$U_k \subset U_i \cap U_j, \text{ and } (\tilde{U}_k, \Gamma_k, \varphi_k),$$

  with

  $$\lambda_{ik} : \tilde{U}_k \to \tilde{U}_i$$

  commuting with the actions of $\Gamma_k$ and $\Gamma_i$.

*Cyclic 4-orbifold*: $\tilde{U}_i = \mathbb{C}^2, \Gamma_i = \mathbb{Z}_m = \langle \psi \rangle \subset U(2)$,

$$\psi(z_1, z_2) = (\psi^{j_1} z_1, \psi^{j_2} z_2), \mathrm{gcd}(j_1, j_2, m) = 1.$$

# Examples of orbifolds

1. $M$ - manifold, $\Gamma$ acts on $M$ properly discontinuously (but not freely). Then $p \in M$ has finite isotropy $\Gamma_p$ and $\gamma \tilde{U}_p \cap \tilde{U}_p = \emptyset, \gamma \notin \Gamma_p$ and $\gamma \tilde{U}_p = \tilde{U}_p, \gamma \in \Gamma_p \implies$
$$U_p = \tilde{U}_p / \Gamma_p$$
yield an orbifold atlas on $X = M/\Gamma$.
2. $S^2$ as a Riemann sphere $\mathbb{C} \cup \infty$ with atlas
$$\varphi_0 : \mathbb{C} \to V_0 = S^2 \setminus \{0\}, \varphi_\infty : \mathbb{C} \to V_\infty : \mathbb{C} \to S^2 \setminus \{\infty\}.$$
with
$$\varphi_0(z) = z^m, \varphi_\infty(w) = w^{-n}, \lambda : \mathbb{C} \to \mathbb{C}, \lambda(z) = (\frac{1}{z})^{\frac{m}{n}}.$$

# Geometric structures on orbifolds (roughly)

$X$ an orbifold with orbifold atlas $\{\tilde{U}_\alpha, \varphi_\alpha, \Gamma_\alpha\}$. An *orbi-tensor* on $X$ is a collection of tensors $T_\alpha$ on each $\tilde{U}_\alpha$ that are $\Gamma_\alpha$-invariant and that agree on intersections. Thus:

- orbi-differential forms $\Omega^*_{orb}(X)$,
- orbi-Riemannian metrics $g$,
- orbi-almost complex structures $J$,
- orbi-Kähler metrics $h = g + i\sqrt{-1}\omega, d\omega = 0$.

# Regular and singular points of an orbifold

For $x \in X$ and a chart $U$ around $x$, the order $m(x)$ of the isotropy subgroup is called *the order of the isotropy*. We condsider 4-orbifolds $X$ whose points are divided into:

- regular, if $m(x) = 1$
- isolated singular ( isotropy) points,
- singular (isotropy) curves (with multiplicities $m_i$), that is, there are $D_i$, with all $x \in D_i$, $m(x) = m_i$.

# Types of local models of 4-orbifolds

$$\xi(z_1, z_2) = (\xi^{j_1} z_1, \xi^{j_2} z_2),$$

$$m_1 = \gcd(j_1, m), m_2 = \gcd(j_2.m) \implies \gcd(m_1, m_2) = 1 \implies$$

$$m_1 m_2 d = m, j_1 = m_1 e_1, j_2 = m_1 e_1, m = m_1 c_1 = m_2 c_2.$$

1. If $\gcd(j_1, m) = 1, \gcd(j_2, m) = 1 \implies (\xi^{j_1} z_1, \xi^{j^2} z_2) \neq (z_1, z_2)$, except $(0, 0) \implies$ the quotient is singular: $\mathbb{C}^2/\mathbb{Z}_m$, $m_1 = 1, m_2 = 1$ $\implies$ $x$ is an isolated singular point.
2. If $m_1, m_2 > 1, d = 1 \implies m = m_1 m_2$, then $\mathbb{Z}_m$-action on $\mathbb{C}^2$ is equivalent to the product action of $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ on $\mathbb{C} \times \mathbb{C}$ (homeomorphic to $\mathbb{C} \times \mathbb{C}$).
3. fixed points in case (2) are non-isolated, they form transversal surfaces $D_1 = \{(z_1, 0)\}, D_2 = \{(0, z_2)\}$.

Type (1) - *non-smooth*, type (2) - *smooth points*. There are more possibilities, but they fall into smooth and non-smooth cases.

# Simpler case: smooth 4-orbifolds

## Definition

We say that a singular point $x \in X$ is *smooth* if $U$ is homeomorphic to a ball in $\mathbb{R}^4$.

## Proposition [Rojo-Muñoz-AT]

*Let $X$ be a smooth oriented 4-manifold with embedded surfaces $D_i$ intersecting transversally, and coefficients $m_i > 1$ such that $\gcd(m_i, m_j) = 1$, if they intersect. Then there exists a smooth orbifold structure on $X$ with isotropy surfaces $D_i$ of multiplicities $m_i$.*

*Proof.* Use the manifold atlas for all points not in $D$. Use the Riemannian metric on a manifold $X$, normal bundle for open $V \subset D$, trivialize $N_D^\varepsilon \cong V \times B_\varepsilon(0) = \tilde{U}$ and define $\mathbb{Z}_m$-action on $\tilde{U}$ by

$$\xi(z_1, z_2) = (z_1, \xi z_2).$$

# Example: smooth symplectic 4-orbifold

**Proposition [Rojo-Muñoz-AT]**

*Let* $(M, \omega)$ *be a symplectic 4-manifold with symplectic surfaces* $D_i$ *intersecting transversally and positively, and coefficients* $m_i > 1$ *such that* $\gcd(m_i, m_j) = 1$ *if* $D_i \cap D_j \neq \emptyset$*. Then there is a smooth symplectic orbifold structure on* $X$*.*

For the proof, use the *manifold* symplectic form and compatible Darboux charts...

# How to construct a non-smooth Kähler orbifold?

- A cyclic singular symplectic (Kähler) orbifold: singular points are isolated (then they are non-smooth);
- Embedded symplectic surfaces $D_i$ intersect nicely, if $D_i = \{z_1, 0\}, D_j = \{0, z_2\}$ in the *Darboux chart*.

**Theorem** [Muñoz] *Let $X$ be a cyclic Kähler 4-orbifold, with a set of singular isolated points $P$. Let $D_i$ be embedded curves intersecting nicely. Take coefficients $m_i > 1$ such that $\gcd(m_i, m_j) = 1$, if $D_i \cap D_j \neq \emptyset$. Then there exists a Kähler orbifold structure $X$ with isotropy surfaces $D_i$ of multiplicities $m_i$, and singular points $x \in P$ of multiplicity $m = d(x) \prod_{i \in I_x} m_i$, where $I_x = \{i \,|\, x \in D_i\}$.*

# Seifert bundles

Let $X$ be a cyclic oriented 4-orbifold. A Seifert bundle over $X$ is an oriented 5-*manifold* $M$ endowed with a smooth $S^1$-action and a continuous map $\pi : M \to X$ such that for an orbifold chart $(U, \tilde{U}, \mathbb{Z}_m, \varphi)$ there is a commutative diagram

$$\begin{array}{ccc} (S^1 \times \tilde{U})/\mathbb{Z}_m & \xrightarrow{\cong} & \pi^{-1}(U) \\ \downarrow & & \downarrow \pi \\ \tilde{U}/\mathbb{Z}_m & \xrightarrow{\cong} & U \end{array}$$

where the action of $\mathbb{Z}_m$ on $S^1$ is by multiplication by $\exp(2\pi i/m)$, and the top diffeomorphism is $S^1$-equivariant.

# Seifert bundles: a condition on local invariants of $X$

Let

$$(X, P, \Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i)$$

For each point $x \in P$ with multiplicity $m = dm_1 m_2$ we have an adapted chart $U \subset \mathbb{C}^2/\mathbb{Z}_m$ with action

$$\exp(2\pi i/m)(z_1, z_2) = (e^{2\pi i j_1/m} z_1, e^{2\pi i j_2/m} z_2),$$

We call $j_x = (m, j_1, j_2)$ the *local invariants* at $x \in P$. Assume that $D_1 = \{(z_1, 0)\}$ is one of the isotropy surfaces with multiplicity $m_1$ and assume that each singular point $x \in P$ lies in a single isotropy surface, if any. The local invariant of $D_1$ is by definition, $j_{D_1} = (m_1, j_2)$, where $j_2$ is considered modulo $m_1$. It is compatibility condition.

# The first Chern class of a Seifert bundle

Given a Seifert bundle $\pi : M \to X$, the order of a stabilizer (in $S^1$) of any point $p$ in the fiber over $x \in X$ is denoted by $m = m(x)$.
For a Seifert bundle $M \to X$ define the first Chern class as follows. Let $l = \operatorname{lcm}(m(x) \,|\, x \in X)$. Denote by $M/l$ the quotient of $M$ by $\mathbb{Z}_l \subset S^1$.
Then $M/l \to X$ is a circle bundle with the first Chern class $c_1(M/l) \in H^2(X, \mathbb{Z})$. Define

$$c_1(M) = \frac{1}{l} c_1(M/l) \in H^2(X, \mathbb{Q}).$$

# How to construct Seifert bundle?

**Theorem.**[Muñoz-AT] *Let $X$ be a cyclic 4-orbifold with a complex structure and $D_i \subset X$ complex curves of $X$ which intersect transversally. Let $m_i > 1$ such that $\gcd(m_i, m_j) = 1$ if $D_i$ and $D_j$ intersect. Suppose that there are given local invariants $(m_i, j_i)$ for each $D_i$ and $j_p$ for every singular point $p$ (and they are compatible). Choose any $0 < b_i < m_i$ such that $j_i b_i \equiv 1 \pmod{m_i}$. Let $B$ be a complex line bundle over $X$. Then there exists a Seifert bundle $M \to X$ with the given local invariants and the first orbifold Chern class*

$$c_1(M) = c_1(B) + \sum_i \frac{b_i}{m_i}[D_i].$$

*Moreover, if $X$ is a Kähler cyclic orbifold and $c_1(M) = [\omega]$ for the orbifold Kähler form, then $M$ is Sasakian.*

# Comparison

- B-W: $S^1 \to M \to (X, \omega)$ corresponding $[\omega] \in H^2(X, \mathbb{Z})$
- VM+AT:

$$(X, P, \Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i), (m, j_1, j_2), (m_1, j_2)...$$

$$c_1(M) = c_1(B) + \sum_i \frac{b_i}{m_i}[D_i]$$

## Conclusion

More flexibility: one wants to choose local invariants $m_i, j_i, b_i$, $P, \Delta$ in an "arbitrary" way, *ensuring, however, $c_1(M)$ to be represented by some Kähler form*!

# Smale-Barden manifolds: the first interesting dimension

A 5-dimensional simply connected manifold $M$ is called a *Smale-Barden manifold*. These manifolds are classified by their second homology group over $\mathbb{Z}$ and the *Barden invariant*.
Write $H_2(M,\mathbb{Z})$ as a direct sum of cyclic groups of prime power order

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus \big( \bigoplus_{p,i} \mathbb{Z}_{p^i}^{c(p^i)} \big),$$

where $k = b_2(M)$. Let $i(M)$ be the smallest integer $j$ such that there is $\alpha \in H_2(M,\mathbb{Z})$ such that $w_2(\alpha) \neq 0$ and $\alpha$ has order $2^j$. $i(M)$ and is called the *Barden invariant*.

# Classification theorem

*Any Smale-Barden manifold is uniquely determined up to diffeomorphism by the data*

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus \Big(\bigoplus_{p,i} \mathbb{Z}_{p^i}^{c(p^i)}\Big),$$

*and $i(M)$.*

Thus, Smale-Barden manifolds are classfied by $b_2(M)$, torsion groups in $H_2(M,\mathbb{Z})$ and $i(M)$.

# Basic cohomology

Let $(M, \xi, \eta, g, J)$ be Sasakian.

## Definition

A differential form $\alpha \in \Omega(M)$ is called *basic*, if

$$\xi \lrcorner \alpha = 0 = \xi \lrcorner d\alpha = 0.$$

The differential complex

$$(\Omega_B(M), d), \Omega_B(M) = \{\alpha \in \Omega(M) \,|\, \alpha \text{ basic}\}$$

yields the *basic cohomology* $H^*_B(M)$.

# Definite Sasakian structures

Let $\mathcal{F}_\xi$ be the characteristic foliation of the Reeb vector field $\xi$ on the Sasakian manifold $M$. Then

$$TM = \mathcal{D} \oplus L_\xi$$

$L_\xi$ al line bundle tangent to leaves of $\mathcal{F}_\xi$. $\mathcal{D}$ is a *complex* vector bundle, hence there are *basic Chern classes* defined as $c_k(\mathcal{F}_\xi) = c_k(\mathcal{D}) \in H_B^{2k}(\mathcal{F}_\xi)$ as elements of the *basic cohomology* of $\mathcal{F}_\xi$.

## Definition

A Sasakian structure is called *negative* if $c_1(\mathcal{F}_\xi)$ can be represented by a negative definite $(1,1)$-form, and it is called *positive*, if it can be represented by a *positive definite* $(1,1)$-form. It is *null*, if $c_1(\mathcal{F}_\xi) = 0$.

# Open problems (Boyer-Galicki, Sasakian Geometry, 2009)

1. (fundamental) *Which Smale-Barden manifolds admit Sasakian structures?*
2. *are there Smale-Barden manifolds which admit K-contact, but do not admit Sasakian structures?*
3. *which simply connected rational homology spheres admit Sasakian structures?*
4. (weaker) *which simply connected rational homology spheres admit both positive and negative Sasakian structures?*
5. *determine when* $\#_k(S^2 \times S^3)$ *admits a negative Sasakian structure?*
6. *determine, when* $\#_k(S^2 \times S^3)$ *admits a null Sasakian structure?*

# Sample of results

In

V. Muñoz, AT,M.Schütt, *Negative Sasakian structures on simply-connected 5-manifolds*, Mathematical Research Letters 29(2022), 1827-1857

we give complete answers to questions 4,5.
In greater detail:
**Theorem 1**. *Any simply connected rational homology sphere admitting positive Sasakian structure admits also a negative Sasakian structure*.

**Theorem 2**. *Any $\#_k(S^2 \times S^3)$ admits negative Sasakian structures*.

# Context

**Theorem** (Kollár) Suppose that a rational homology sphere admits a positive Sasakian structure. Then $M$ is spin, and $H_2(M, \mathbb{Z})$ is one of the following:

$$0, \mathbb{Z}_m^2, \mathbb{Z}_5^4, \mathbb{Z}_4^4, \mathbb{Z}_3^4, \mathbb{Z}_3^6, \mathbb{Z}_3^8, \mathbb{Z}_2^{2n}$$

where $n > 0, m \geq 2$, $m$ not divisible by 30. Conversely, all these cases do occur.

**Theorem** (Kollár) *Let $M$ be a Sasakian 5- manifold with positive Sasakian structure. Then, a positive Ricci curvature metric (resp. Einstein metric) on $X$ can be lifted to a positive Ricci curvature metric (resp. Einstein metric) on $M$. Moreover, the lifted metric is also Sasakian (thus, it is Sasakian-Einstein)*.

# Context, contd.

Kollár’s theorems yielded (2005) new examples of Einstein metrics on simply connected homology spheres (and even on spheres).

Negative Sasakian structures do not yield Sasaki-Einstein metrics, but yield $\eta$-Einstein metrics:

$$\mathrm{Ric}_g = \lambda g + \nu \eta \otimes \eta$$

(a "transversal version of the Aubin-Yau theorem") (Boyer-Galicki).

# Realization problem

To answer Question 4 of Boyer and Galicki, one needs to construct a manifold $M$ with $b_2(M) = 0$, and torsion groups in the second homology as in Kollár's theorem and with a negative Sasakian structure.

# General scheme of proof

1. A quasi-regular Sasakian structure arises from *Seifert bundle* $M \to X$, where $X$ is a *Kähler orbifold* (Rukimbira),
2. There is a method of "Kollár's type" of calculating $H_*(M, \mathbb{Z})$ from $H_*(X, \mathbb{Z})$, in particular $H_2(M), \mathbb{Z})$,
3. CONSTRUCT a Kähler orbifold $X$ and a Seifert bundle $M \to X$ with such $H_*(X)$ that $H_2(M)$ has prescribed $b_2 = 0$, or $b_2 = k$, torsion groups and $i(M)$, AND $\pi_1(M) = 1$. If such $X$ is constructed, we get $M$ as the rational homology sphere or $\#k(S^2 \times S^3)$
4. Also, EXPRESS in these terms the negativity condition, and check if it is compatible with the other data.

# (1): a method of construction of Kähler orbifolds

**Theorem**[Muñoz-Schütt-AT] *Let $X$ be a smooth complex surface containing a chain of smooth rational curves $E_1, \ldots, E_l$ of self-intersections $-b_1, -b_2, \ldots, -b_l$, with all $b_i \geq 2$, intersecting transversally. Let $\pi : X \to \bar{X}$ be the contraction of $E = E_1 \cup \ldots \cup E_l$. Then $\bar{X}$ has a cyclic singularity at $p = \pi(E)$, with an action $\mathbb{Z}_m$ on $\mathbb{C}^2$ given by $(z_1, z_2) \to (\xi z_1, \xi^r z_2), 0 < r < m$. Moreover, if $D$ is a curve intersecting transversally a tail of the chain (that is, either $E_1$ or $E_l$ but not $E_1 \cap E_l$), then the push down curve $\bar{D} = \pi(D)$ is an orbismooth curve in $\bar{X}$.*

Note: the theorem is derived from results in

P. Popescu-Pampu, *Introduction to Jung's method of resolution of singularities*, 2011.

# (2): Kollár's type results on topology of Seifert bundles

**Theorem**[Muñoz-AT] *Suppose that* $\pi : M \to X$ *is a quasi-regular Seifert bundle over a cyclic orbifold* $X$ *with isotropy surfaces* $D_i$ *and set of singular points* $P$*. Let* $\mu = \text{lcm}(m_i)$*. Then* $H_1(M, \mathbb{Z}) = 0$ *if and only if*

1. $H_1(X, \mathbb{Z}) = 0$,
2. $H^2(X, \mathbb{Z}) \to \oplus H^2(D_i, \mathbb{Z}_{m_i})$ *is onto,*
3. $c_1(M/\mu) \in H^2(X - P, \mathbb{Z})$ *is primitive.*

*Moreover,* $H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus (\bigoplus_i \mathbb{Z}_{m_i}^{2g_i})$*,* $g_i =$ *genus of* $D_i$*,* $k + 1 = b_2(X)$*.*

# (4): Canonical class and negativity condition

Let $X$ is a complex surface (orbifold). A canonical line bundle $K_X$ is $\Lambda^2 T^* X$ (a determinant line bundle oven $X$). An *orbifold canonical class* is

$$K_X^{orb} = K_X + \Delta$$

where $\Delta$ is a branch divisor.

**Theorem** [Boyer-Galicki] *A manifold $M$ admits a quasi-regular negative Sasakian structure if and only if the base $X$ of the Seifert bundle $M \to X$ has the property that the canonical class $K_X^{orb}$ is ample.*

**Ampleness (Nakai-Moishezon criterion)** A line bundle $L \to X$ is ample, if and only if

$$c_1^2(L) > 0 \text{ and } L \cdot D > 0$$

for any effective divisor $D$.

# What to do "in practice" (Example of Theorem 1)

1. Fix some data for rational homology sphere, that is, torsion group, say $\mathbb{Z}_2^{2n}$
2. find smooth algebraic surface $Y$ with a chain of curves $E_i$ as in the general construction, and blow down this configurattion, getting an orbifold $X$, and a branch divisor $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i$ (assign appropriate $m_i$),
3. using algebraic geometry formulas (e.g. "adjunction formula"), calculate $K_X^{orb}$, the *branch divisor should ensure the ampleness* of $K_X^{orb}$,
4. construct a Seifert bundle choosing

$$c_1(M) = c_1(B) + \sum_i \frac{b_i}{m_i}[D_i]$$

   in way to ensure the primitivity condition and the "correct homology", that is, $b_2(M) = 0$ and the torsion $\mathbb{Z}_2^{2n}$, and $c_1(M) = [\omega]$ for some orbifold Kähler form $[\omega]$ on $X$.
5. show that $\pi_1(M) = 1$ under the choices made.

# Conclusion

The key points are:

- to guess what smooth algebraic surface to take,
- what chain of curves $E_i$ should be taken, and calculate the rest,
- the flexibility is ensured by the choice of local invariants and the branch divisor.

# Example of the choice: $H_2(M) = \mathbb{Z}_2^{2n}$

1. The algebraic surface: the Hirzebruch surface $\mathbb{F}_n$,
2. the chain: only one curve - infinity section $E_\infty$, $X$ is a blowdown of $\mathbb{F}_n$ along $E_\infty$,
3. the branch divisor $\Delta = (1 - \frac{1}{m})D + \sum_{i=1}^{s}(1 - \frac{1}{m_i})D_i$, where

$$D = C + \beta E_0, D_i = E_{\sigma_i}$$

$m_1, ..., m_s$ are pairwise coprime, $m_i$ a coprime to $m$ and to $n$, $\beta \geq 1$, $C$ is a fiber of the fibration $\mathbb{F} \to \mathbb{C}^1$.

# Notation and formulas

1. $\mathbb{F}_n = P(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1})$,
2. "infinity section": $\sigma_\infty : \mathbb{C}P^1 \to \mathcal{O}_{\mathbb{C}P^1}(n)$ gives the image $E_\infty$ of $(\sigma, 0)$, where $\sigma : \mathbb{C}P^1 \to \mathcal{O}_{\mathbb{C}P^1}(n)$ is a holomorphic section
3. section $\sigma$ defines the image $C$ of $(\sigma, 1)$ in $\mathbb{F}_n$.

Intersection numbers: $E_\infty \cdot E_\infty = -n$, $E_i \cdot E_i = n$, $C^2 = 0$, $E_0 \cdot C = 1$.

Adjunction formula for orbifolds:
$K_X^{orb} \cdot D + D^2 = -\chi^{orb}(D) = 2g(D) - 2 + \sum_p (1 - \frac{1}{n_p})$, where $n_p$ are the orders of isolated singularities.

Genus formula: $D^2 + K_X \cdot D = 2g(D) - 2$.

# Verification of ampleness

One calculates first $K_X$ using adjunction formlua to get

$$K_X = -(n+2)C$$

and then

$$K_X^{orb} = K_X + \Delta =$$

$$(-(n+2) + (1 - \frac{1}{m})(1 + \beta n) + n \sum_{i=1}^{n} (1 - \frac{1}{m_i}))C.$$

The ampleness is equivalent to the positivity of the coefficients in this expression.

# Null Sasakian structures

In

A. Cañaz, V. Muõz, M. Schütt, AT, *Quasi-regular Sasakian and K-contact structures on Smale-Barden manifolds*, Revista Mat. Iberoam. 38(2022), 1027-1050

we "did the same" for Seifert bundles over orbifold K3 surfaces and got the full answer to Question 6 of Boyer and Galicki.

# Some references


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# A Survey

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