Nichols algebras over groups

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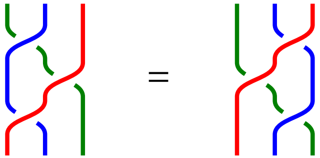
Cohomology in algebra, geometry, physics and statistics
Institute of Mathematics of ASCR – October 2025

Braided vector spaces

Let (V,c) be a braided vector space, that is a K-vector space V with a bijective linear map $c \in \mathbf{GL}(V \otimes V)$ that satisfies the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$



- If V has dimension n, say with basis $\{v_1, \ldots, v_n\}$, then the tensor product $V \otimes V$ is the vector space of dimension n^2 with basis

$$\{v_i \otimes v_j : 1 \leq i,j \leq n\}.$$

- If $A \in K^{m \times n}$ y $B \in K^{p \times q}$, the Kronecker product of $A = (a_{ij})$ and B is the matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

A concrete example

The matrix

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

satisfies

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Another example

Let V be a vector space with basis $\{x_1, x_2, \ldots, x_\theta\}$. Then

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i,$$

where $q_{ij} \in K^\times$, is a braiding (of diagonal type).

Yet another example

Let G be a group and X be a union of conjugacy classes of G. Let V be a K-vector space with basis X. Then

$$c(x \otimes y) = q_{xy} xyx^{-1} \otimes x,$$

where $q_{xy} \in K^{\times}$ is a certain collection of scalars, is a braiding (of group type).

Braid groups

The braid group $\mathbb{B}_n$ has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad 1 \leq i \leq n-2,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad i - j \geq 2.$$

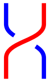
If (V, c) is a braided vector space, then

$$\rho \colon \mathbb{B}_n \to \mathbf{GL}(V^{\otimes n}), \quad \sigma_i \mapsto c_i,$$

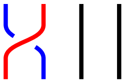
is a group homomorphism, where

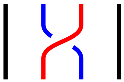
$$c_k = \mathrm{id}^{\otimes(k-1)} \otimes c \otimes \mathrm{id}^{\otimes(n-k-1)}.$$

For example, let $n = 4$. If we represent the braiding c by the diagram



then the maps c_1 and c_2 are given by





respectively.

A braided vector space (V, c) gives a special type of algebra called the Nichols algebra $\mathscr{B}(V, c)$.

Nichols algebras

The Nichols algebra of (V, c) is constructed as a quotient of the tensor algebra of V:

$$\mathscr{B}(V,c) = K \oplus V \oplus \bigoplus_{n \geq 2} V^{\otimes n} / \ker \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the quantum symmetrizer. For example:

$$\begin{aligned}
\mathfrak{S}_2 &= \mathrm{id} + c, \\
\mathfrak{S}_3 &= \mathrm{id} + c_1 + c_2 + c_1 c_2 + c_2 c_1 + c_1 c_2 c_1, \\
&\vdots \\
\mathfrak{S}_{n+1} &= (\mathrm{id} \otimes \mathfrak{S}_n)(\mathrm{id} + c_1 + c_1 c_2 + \cdots + c_1 c_2 \cdots c_n).
\end{aligned}$$

Some well-known examples of Nichols algebras:

- (V, flip) gives the symmetric algebra.
- $(V, -\text{flip})$ gives the exterior algebra.

Nichols algebras (also known as Fock spaces) were rediscovered several times: Nichols, Woronowicz, Lusztig, Andruskiewitsch–Schneider, Majid...

Nichols algebras have more structure:

- They are braided Hopf algebras.
- They are graded by non-negative integers:

$$\mathscr{B}(V) = \bigoplus_{n \geq 0} \mathscr{B}_n(V) = K \oplus V \oplus \mathscr{B}_2(V) \oplus \mathscr{B}_3(V) \oplus \cdots$$

The Hilbert series of a Nichols algebra

$$\mathscr{B}(V) = K \oplus V \oplus \mathscr{B}_2(V) \oplus \mathscr{B}_3(V) \oplus \cdots$$

is the (formal) series

$$H(t) = 1 + (\dim V)t + \sum_{n \geq 2} (\dim \mathscr{B}_n(V))t^n.$$

If $\dim \mathscr{B}(V) < \infty$, then $H(t)$ is a polynomial.

Problem

Classify finite-dimensional Nichols algebras.

For applications, the interesting Nichols algebras all come from groups. Which braided vector spaces should we consider?

Yetter-Drinfeld modules (over groups)

Let G be a group. A Yetter-Drinfeld module V over G is a G-graded KG-module

$$V = \oplus_{g \in G} V_g$$

such that

$$g \cdot V_h \subseteq V_{ghg^{-1}}$$

for all $g, h \in G$.

Yetter–Drinfeld modules were discovered in 1949 by Whitehead.

John Whitehead (1904–1960).

Yetter–Drinfeld modules were rediscovered several years later in connection to quantum groups and solutions to the Yang–Baxter equation.

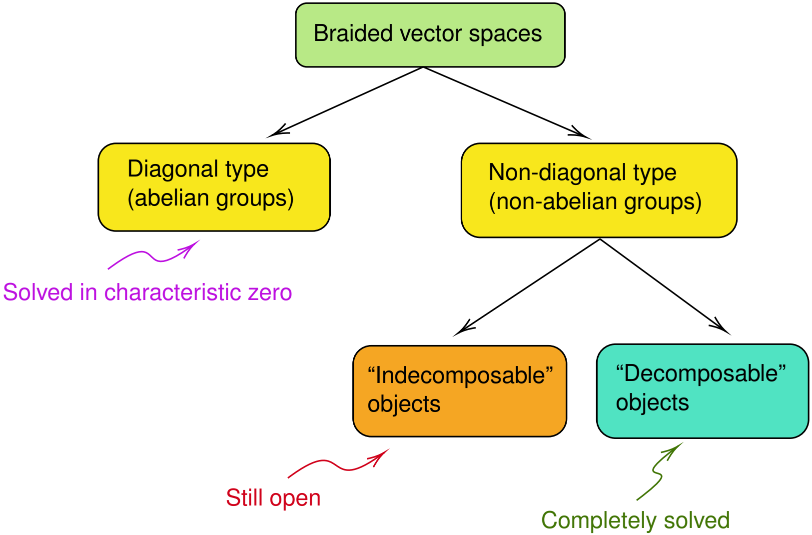
Fact:
A Yetter-Drinfeld module V is a braided vector space:

$$c(v \otimes w) = g \cdot w \otimes v,$$

where $v \in V_g$ and $w \in V$.

Moreover, the category ${}^{KG}_{KG}\mathcal{YD}$ of Yetter-Drinfeld modules over KG is a braided tensor category.

The state of the art



Braided vector spaces of diagonal type

A braided vector space V is of diagonal type if there exists a basis $\{v_1, \cdots, v_\theta\}$ of V such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad q_{ij} \in K^\times.$$

Nichols algebras of braided vector spaces of diagonal type have many interesting properties and applications.

Complex finite-dimensional Nichols algebra of diagonal type:

- Classified by Heckenberger.
- Generators and relations: Angiono.
- Applications to Hopf algebras: Andruskiewitsch and Schneider's classification.
- Applications to physics: Semikhatov, Lentner.
- Applications to the Etingof-–Ostrik conjecture: Andruskiewitsch, Angiono, Pevtsova, Witherspoon, Jaklitsch, Nguyen, Oswald, Plavnik, Shepler, Wang.

What about non-diagonal Nichols algebras?

This is important to study combinatorial Schubert calculus and pointed Hopf algebras with non-abelian coradical.

We will show some concrete examples, over “indecomposable” braided vector spaces.

Example 1

This example goes back independently to Fomin–Kirillov and Milinski–Schneider.

Let X be the conjugacy class of (12) in the symmetric group $\mathbb{S}_3$. Let V be the complex vector space with basis

$$a = v_{(12)}, \quad b = v_{(13)}, \quad c = v_{(23)}$$

and

$$c(v_g \otimes v_h) = -v_{ghg^{-1}} \otimes v_g, \qquad g, h \in X.$$

This braided vector space can be realized as a Yetter–Drinfeld module over $\mathbb{S}_3$.

Example 1

The relations in degree two are:

$$
\begin{aligned}
0 &= a^2, \\
0 &= b^2, \\
0 &= c^2, \\
0 &= ab + bc + ca, \\
0 &= ac + ba + cb.
\end{aligned}
$$

One can prove that $\dim \mathscr{B}(V) = 12$ and

$$\{1, a, b, c, ab, ac, ba, bc, aba, abc, bac, abac\}$$

is a basis. Moreover, the Hilbert series of $\mathscr{B}(V)$ is

$$1 + 3t + 4t^2 + 3t^3 + t^4.$$

After discussing a case which yields a finite-dimensional Nichols algebra, let me mention a related important family of braided vector spaces.

Consider V with basis

$$a = v_{(12)}, \quad b = v_{(13)}, \quad c = v_{(23)}$$

and braiding

$$c(v_g \otimes v_h) = \omega v_{ghg^{-1}} \otimes v_g, \qquad g, h \in X,$$

where ω is a primitive root of unity of order > 2.

For some time, it remained an open problem whether the corresponding Nichols algebra is finite-dimensional.

This problem was solved only recently, in joint work with Heckenberger and Meir: $\dim V = \infty$.

Example 2

This example goes back to Graña.

Let X be the conjugacy class of (123) in $\mathbb{A}_4$ and V the complex braided vector space with basis

$$a = v_{(243)}, \quad b = v_{(123)}, \quad c = v_{(134)}, \quad d = v_{(142)}$$

and braiding

$$c(v_g \otimes v_h) = -v_{ghg^{-1}} \otimes v_g, \qquad g, h \in X.$$

This braided vector space cannot be realized as a Yetter–Drinfeld module over $\mathbb{A}_4$, but over the group $C_2 \times \mathbb{A}_4$.

Example 2

The relations in degree two are:

$$0 = a^2,$$
$$0 = b^2,$$
$$0 = c^2,$$
$$0 = d^2,$$
$$0 = ba + db + ad,$$
$$0 = ca + bc + ab,$$
$$0 = da + cd + ac,$$
$$0 = cb + dc + bd,$$

and there is one relation in degree six:

$$0 = cbacba + bacbac + acbacb.$$

Example 2

One can prove that $\dim \mathscr{B}(V) = 72$. The Hilbert series of $\mathscr{B}(V)$ is

$$1 + 4t + 8t^2 + 11t^3 + 12t^4 + 12t^5 + 11t^6 + 8t^7 + 4t^8 + t^9.$$

Now let me turn to a different example. Here, too, there was an open problem: what happens with the Nichols algebras associated to the group $\mathbb{A}_4$?

We consider the braided vector space with basis

$$a = v_{(243)}, \quad b = v_{(123)}, \quad c = v_{(134)}, \quad d = v_{(142)}$$

and braiding

$$c(v_g \otimes v_h) = \omega v_{ghg^{-1}} \otimes v_g, \qquad g, h \in X.$$

where ω is a primitive cubic root of one.

Is $\dim \mathscr{B}(V) < \infty$?

No. This problem was solved only recently, in joint work with Andruskiewitsch and Heckenberger.

An open problem: Fomin–Kirillov algebras

For $n \geq 3$, let X_n be the conjugacy class of (12) in the symmetric group $\mathbb{S}_n$. Let V_n be the complex vector space with basis

$$\{v_g : g \in X_n\}$$

and

$$c(v_g \otimes v_h) = -v_{ghg^{-1}} \otimes v_g.$$

Question

When is $\dim \mathscr{B}(V_n) = \infty$?

Fact:

$\mathscr{B}(V_n)$ is finite-dimensional for $n \in \{3,4,5\}$.

n	$\dim V_n$	$\dim \mathscr{B}(V_n)$
3	3	12
4	6	576
5	10	8294400

Conjectures

- $\dim \mathscr{B}(V_n) = \infty$ for $n \geq 6$.
- $\mathscr{B}(V_n)$ is quadratic.

So far only few examples of finite-dimensional Nichols algebras over “indecomposable” braided vector spaces of group type are known!

Known examples

This is the list of known finite-dimensional examples over "inde-composable" complex braided vector spaces.

$\dim V$	$\dim \mathscr{B}(V)$	group
3	12	$\mathbb{S}_3$
4	72	$C_2 \times \mathbb{A}_4$
4	5184	$SL_2(3)$
6	576	$\mathbb{S}_4$
6	576	$\mathbb{S}_4$
6	576	$\mathbb{S}_4$
5	1280	$C_5 \rtimes C_4$
5	1280	$C_5 \rtimes C_4$
7	326592	$C_7 \rtimes C_6$
7	326592	$C_7 \rtimes C_6$
10	8294400	$\mathbb{S}_5$
10	8294400	$\mathbb{S}_5$

Question

Are there other finite-dimensional Nichols algebras?

Let us describe the full solution to the “decomposable” case now.

What does it mean “decomposable”?

We first start with the case of two irreducible summands.

Remark (Graña)

If $c_{W,V}c_{V,W} = \mathrm{id}_{V\otimes W}$ then

$$\mathscr{B}(V \oplus W) \simeq \mathscr{B}(V) \otimes \mathscr{B}(W)$$

as graded vector spaces.

The remark implies that in order to be in the “decomposable” case one needs to assume that

$$c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}.$$

Technical definition:
The support of a Yetter-Drinfeld module

$$V = \oplus_{g \in G} V_g$$

is the set

$$\mathrm{supp} V = \{g \in G : V_g \neq 0\}.$$

Fact:
$\mathrm{supp}(V)$ is a union of conjugacy classes of G.

Theorem (with Heckenberger)

Let G be a non-abelian group, and V and W be two absolutely irreducible Yetter-Drinfeld modules over KG. Assume that

- G is generated by the support of $V \oplus W$,
- $c_{W,V} c_{V,W} \neq \mathrm{id}_{V \otimes W}$, and
- $\dim \mathscr{B}(V \oplus W) < \infty$.

Then G is either an epimorphic image of a certain central extension T of the group $\mathbf{SL}_2(3)$, or an epimorphic image of a certain central extension of the dihedral group of order $2n$ for $n \in \{2, 3, 4\}$.

Non-diagonal type: the "decomposable" case

The theorem has deep consequences. One obtains:

- The structure of the braided vector spaces V and W.
- The dimension of $\mathscr{B}(V \oplus W)$.

Theorem (with Heckenberger)

Let G be a non-abelian group, and V and W be two irreducible Yetter-Drinfeld modules over $\mathbb{C}G$. Assume that

- G is generated by the support of $V \oplus W$,
- $c_{W,V} c_{V,W} \neq \mathrm{id}_{V \otimes W}$, and
- $\dim \mathscr{B}(V \oplus W) < \infty$.

Then $\mathscr{B}(V \oplus W)$ is one of following Nichols algebras:

$\dim(V \oplus W)$	$\dim \mathscr{B}(V \oplus W)$
4	64
4 or 5	10368
5	2304
5	80621568
6	262144

An example: the group T

Let us show one of the examples we found (over the complex numbers).

The group T can be presented by generators z, x_1, x_2, x_3, x_4 and relations

$$zx_i = x_i z, \quad i \in \{1, 2, 3, 4\},$$

and

$$x_1 x_2 = x_4 x_1 = x_2 x_4,$$
$$x_1 x_3 = x_2 x_1 = x_3 x_2,$$
$$x_2 x_3 = x_4 x_2 = x_3 x_4,$$
$$x_1 x_4 = x_3 x_1 = x_4 x_3.$$

An example: the module V

Let G be a non-abelian epimorphic image of the group T. We show the structure of the modules V and W.

How does V look like? Let ρ be a character of the centralizer $G^z = G$ and $v \in V_z \setminus \{0\}$. Then $\{v\}$ is basis of V and the action of G on V is given by

$$zv = \rho(z)v, \quad x_i v = \rho(x_1)v \quad \text{for all } i \in \{1,2,3,4\}.$$

An example: the module W

How does W look like? Let σ be a character of $G^{x_1} = \langle x_1, x_2x_3, z\rangle$ with $\sigma(x_1) = -1$ and $\sigma(x_2x_3) = 1$. Let $w_1 \in W_{x_1}$ be such that $w_1 \neq 0$. Then the vectors

$$w_1,\ w_2 = -x_4w_1,\ w_3 = -x_2w_1,\ w_4 = -x_3w_1$$

form a basis of W. The degrees of these vectors are x_1, x_2, x_3 and x_4, respectively. The action of G on W is given by the following table:

W	w_1	w_2	w_3	w_4
x_1	$-w_1$	$-w_4$	$-w_2$	$-w_3$
x_2	$-w_3$	$-w_2$	$-w_4$	$-w_1$
x_3	$-w_4$	$-w_1$	$-w_3$	$-w_2$
x_4	$-w_2$	$-w_3$	$-w_1$	$-w_4$
z	$\sigma(z)w_1$	$\sigma(z)w_2$	$\sigma(z)w_3$	$\sigma(z)w_4$

An example: the dimension

Assume further that

$$(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0, \quad \rho(x_1 z)\sigma(z) = 1.$$

Then

$$\dim \mathscr{B}(V \oplus W) = 6^3\, 72^3 = 80621568.$$

The theorem holds for arbitrary characteristic!

The Nichols algebras of the classification

$\dim(V \oplus W)$	$\dim \mathscr{B}(V \oplus W)$	characteristic
4	64	
4	1296	3
4 or 5	10368	$\neq 2, 3$
4 or 5	5184	2
4 or 5	1152	3
4 or 5	2239488	2
5	2304	
5	80621568	$\neq 2$
5	1259712	2
6	262144	$\neq 2$
6	65536	2

We now study the case of at least three irreducible summands.

To be in the "decomposable" case, we need to assume that

$$M = (M_1, \dots, M_\theta)$$

is connected, i.e. $M_1 \oplus \cdots \oplus M_\theta$ admits no decomposition

$$M_1 \oplus \cdots \oplus M_\theta = M' \oplus M''$$

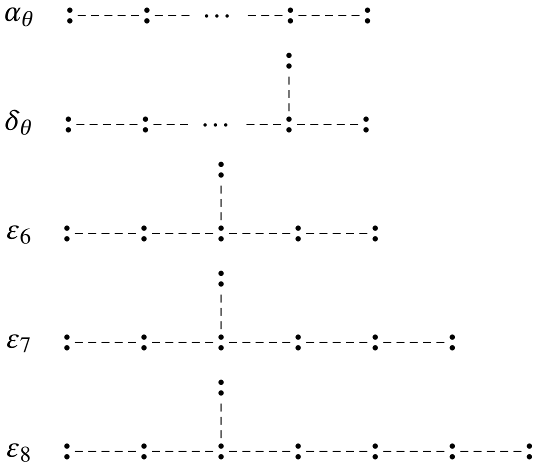
as Yetter-Drinfeld modules over G with $M' \neq 0$, $M'' \neq 0$ and

$$c_{M'',M'} c_{M',M''} = \mathrm{id}.$$

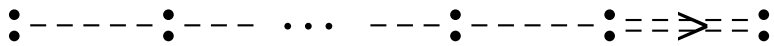
We need to introduce the following terminology.

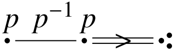
Skeletons (of finite type). A skeleton (of finite-type) is a decorated Dynkin diagram (of finite-type) that encodes the structure of the Yetter-Drinfeld module.

The following are the simply-laced skeleton of finite-type (i.e. Dynkin types ADE):

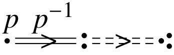


The other skeletons of finite type are:

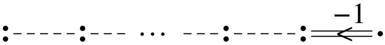
β_θ  $\mathrm{char} K = 3$

β'_3 

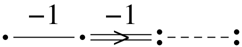
$(3)_{-p} = 0$

β''_3 

$(3)_{-p} = 0$

γ_θ 

$\mathrm{char} K \neq 2$

φ_4 

$\mathrm{char} K \neq 2$

Here $(n)_t = 1 + t + \cdots + t^{n-1}$.

Theorem (with Heckenberger)

Let $\theta \geq 3$, G be a non-abelian group and

$$M = (M_1, \dots, M_\theta)$$

be a connected tuple of absolutely irreducible Yetter-Drinfeld modules over KG. Then $\dim \mathscr{B}(M_1 \oplus \cdots \oplus M_\theta) < \infty$ if and only if M has a skeleton of finite-type.

The theorem gives the dimensions of

$$\mathscr{B}(M) = \mathscr{B}(M_1 \oplus \cdots \oplus M_\theta)$$

and the structure of the M_i can be obtained from the skeletons of finite type.

Example:
In the case where M has a simply-laced skeleton of finite type (Dynkin type ADE), the dimensions of the Nichols algebras in the classification are

$\dim \mathscr{B}(M)$	$4^{\theta(\theta+1)/2}$	$4^{\theta(\theta-1)}$	4^{36}	4^{63}	4^{120}
skeleton	α_θ	δ_θ	ε_6	ε_7	ε_8

These theorems have strong applications to the “indecomposable” case.

Strategy

Let $V \in {}^{KG}_{KG}\mathcal{YD}$. We want to prove that $\dim \mathscr{B}(V) = \infty$. Find a braided subspace $W \subseteq V$ such that $\dim \mathscr{B}(W) = \infty$. Since

$$\mathscr{B}(W) \subseteq \mathscr{B}(V).$$

it follows that $\dim \mathscr{B}(V) = \infty$.

For example, W could be “decomposable” with $\dim \mathscr{B}(W) = \infty$ by some of the theorems mentioned before.

Theorem (with Andruskiewitsch, Fantino and Graña)

Let $n \geq 5$ and $G = \mathbb{A}_n$. If $0 \neq V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.

A similar result is valid for sporadic simple groups.

Theorem (with Andruskiewitsch, Fantino and Graña)

Let G be a finite sporadic simple group. If $G \notin \{Fi_{22}, B, M\}$ and $0 \neq V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.

Question

Let G be the Fischer group Fi_{22}, the Baby Monster B or the Monster M, and let $0 \neq V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$. Is $\dim \mathscr{B}(V) = \infty$?

Several results concerning Nichols algebras over finite simple groups of Lie type were found by Andruskiewitsch, Carnovale, Costantini, García.

Conjecture

Let G be a finite non-abelian simple group. If $0 \neq V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathscr{YD}$, then $\dim \mathscr{B}(V) = \infty$.

Let us now see some classification results that use the theorems mentioned before.

Theorem (with Heckenberger and Meir)

Let G be a non-abelian group and $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathscr{YD}$ be an irreducible of prime dimension. Assume that $\operatorname{supp} V$ generates G. Then $\dim \mathscr{B}(V) < \infty$ if and only if $\mathscr{B}(V)$ is one of the following Nichols algebras:

$\dim V$	$\dim \mathscr{B}(V)$
3	12
5	1280
5	1280
7	326592
7	326592

The tools used to previous the previous theorem can be push forward to obtain far more general theorems on the structure of finite-dimensional Nichols algebras.

Theorem (with Andruskiewitsch and Heckenberger)

Let G be a non-cyclic solvable group and $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$. Assume that $\operatorname{supp} V$ generates G. If $\dim \mathscr{B}(V) < \infty$, then V is irreducible and $\mathscr{B}(V)$ is one of the following "known" Nichols algebras:

$\dim V$	$\dim \mathscr{B}(V)$	
3	12	
4	72	
4	5184	
5	1280	two algebras
6	576	three algebras
7	326592	two algebras

The theorem has applications to Hopf algebras (e.g. yields some sort of Feit–Thompson theorem for finite-dimensional pointed Hopf algebras) and to study other Nichols algebras.

Thanks!