

# About graded coverings of supermanifolds and their applications

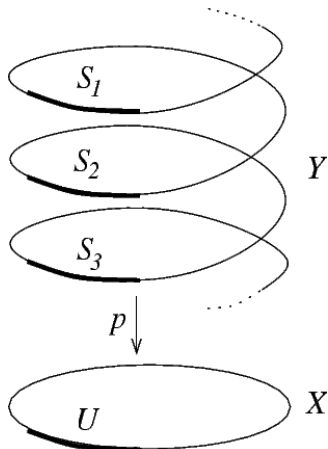
Elizaveta Vishnyakova

Universidade Federal de Minas Gerais

April 1, 2026

Supported by CNPq and FAPEMIG

# Coverings in geometry



$$p : \mathbb{R} \rightarrow S^1, \text{ where } p(t) = e^{2\pi it}.$$

# Properties of a universal covering

- For any smooth connected manifold  $X$  **there exists a simply connected covering**  $p : Y \rightarrow X$ , which is unique up to isomorphism.
- Let  $X'$  be connected, simply connected. Any smooth map  $f : X' \rightarrow X$ , **factors through**  $Y$ :

$$\begin{array}{ccc} & & Y \\ & \nearrow \exists F & \downarrow p \\ X' & \xrightarrow{f} & X \end{array}$$

A covering of  $X$  in the **category of simply connected manifolds**.

Jinzhong Xu, Flat Covers of Modules. Lecture Notes in Mathematics, 1996.

Let  $R$  be a ring and  $\mathcal{F}$  be a family of left  $R$ -modules.

**Definition.** Let  $M$  be an  $R$ -module. A surjective morphism  $p : C \rightarrow M$ , where  $C \in \mathcal{F}$ , is called an  $\mathcal{F}$ -**cover** of  $M$  if:

- a) Any morphism  $f : N \rightarrow M$ , where  $N \in \mathcal{F}$ , is factored through  $p$

$$\begin{array}{ccc} & & C \\ & \nearrow F & \downarrow p \\ N & \xrightarrow{f} & M \end{array}$$

- b) If  $\phi : C \rightarrow C$  such that  $p \circ \phi = p$ , then  $\phi$  is automorphism.

**Example 1. (Torsion-free cover.)** Over an integral domain, every module  $M$  has a torsion-free cover  $p : C \rightarrow M$ .

$$C \in \mathbf{L} = \{\text{torsion-free modules}\} \subset \{\text{all modules}\}.$$

**Example 2. (Flat cover.)** Flat cover conjecture (Enochs 1981).

**For a general ring, every module  $M$  has a flat cover  $C$ .**

The conjecture resolved positively simultaneously by Bican, El Bashir and Enochs (2001).

# Graded coverings of supermanifolds

# Overview

## Graded covering for a supermanifold

Two categories:

- 1 **Supermanifolds:** the objects and morphisms are  $\mathbb{Z}_2$ -graded.
- 2 **Graded manifolds:** the objects and morphisms are (non-negatively)  $\mathbb{Z}$ -graded. They are also called  $\mathbb{N}$ -graded manifolds.

# Overview

## Graded covering for a supermanifold

Two categories:

- 1 **Supermanifolds:** the objects and morphisms are  $\mathbb{Z}_2$ -graded.
- 2 **Graded manifolds:** the objects and morphisms are (non-negatively)  $\mathbb{Z}$ -graded. They are also called  $\mathbb{N}$ -graded manifolds.

**Idea:** To define a covering of a supermanifold in the category of graded manifolds: **graded covering** for a supermanifold, which has to satisfy universal properties.

**Theorem** (E.V.) For any supermanifold there exists a graded covering.

## Briefly about graded manifolds

**Graded manifolds:** the objects and morphisms are  $\mathbb{Z}$ -graded. In general,  $H$ -graded, where  $H$  is a finitely generated abelian group.

## Briefly about graded manifolds

**Graded manifolds:** the objects and morphisms are  $\mathbb{Z}$ -graded. In general,  $H$ -graded, where  $H$  is a finitely generated abelian group.

A **graded manifold**  $\mathcal{N}$  is locally isomorphic to a graded domain.

**Graded domain.** Local coordinates:

$$x_1^0, \dots, x_{n_0}^0, \quad \xi_1^k, \dots, \xi_{n_k}^k, \quad k \in \mathbb{N},$$

with

$$|x_i^0| = \bar{0}, \quad |\xi_j^k| = \bar{k} \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}.$$

## Graded domain

Consider the following  $\mathbb{Z}$ -graded vector superspace  $V$

$$V = \bigoplus_{i=0}^n V_i, \quad V = V_{\bar{0}} \oplus V_{\bar{1}}, \quad V_{\bar{0}} = \bigoplus_{i=2q} V_i, \quad V_{\bar{1}} = \bigoplus_{i=2q+1} V_i,$$

where  $\dim V_i < \infty$  for any  $i \in \mathbb{Z}$ .

Let  $S^*(V)$  be the **super-symmetric algebra of  $V$** .

Let  $\mathcal{V}_0 \subset V_0^*$  be an open set and  $\mathcal{O}_{\mathcal{V}}$  be the sheaf generated by the following presheaf

$$\mathcal{F}_{\mathcal{V}_0} \otimes_{S^*(V_0)} S^*(V),$$

where  $\mathcal{F}_{\mathcal{V}_0}$  is the sheaf of smooth or holomorphic functions on  $\mathcal{V}_0$ . We put  $\mathcal{V} = (\mathcal{V}_0, \mathcal{O}_{\mathcal{V}})$ .

### Definition

The ringed space  $\mathcal{V}$  is called a *graded domain of degree  $n$  and of dimension  $\{n_i\}$* , where  $n_i := \dim V_i$ ,  $i = 0, \dots, n$ .

Let  $\mathcal{V}_0 \subset V_0^*$  be an open set and  $\mathcal{O}_{\mathcal{V}}$  be the sheaf generated by the following presheaf

$$\mathcal{F}_{\mathcal{V}_0} \otimes_{S^*(V_0)} S^*(V),$$

where  $\mathcal{F}_{\mathcal{V}_0}$  is the sheaf of smooth or holomorphic functions on  $\mathcal{V}_0$ . We put  $\mathcal{V} = (\mathcal{V}_0, \mathcal{O}_{\mathcal{V}})$ .

### Definition

The ringed space  $\mathcal{V}$  is called a *graded domain of degree  $n$  and of dimension  $\{n_i\}$* , where  $n_i := \dim V_i$ ,  $i = 0, \dots, n$ .

### Definition

A *graded manifold of degree  $n$  and of dimension  $\{n_i\}$* ,  $i = 0, \dots, n$ , is a  $\mathbb{Z}$ -graded ringed space  $\mathcal{N} = (\mathcal{N}_0, \mathcal{O}_{\mathcal{N}})$ , that is locally isomorphic to a graded domain of degree  $n$  and of dimension  $\{n_i\}$ ,  $i = 0, \dots, n$ .

# Briefly about graded manifolds

Graded manifolds are used for instance in mathematical physics and Poisson geometry.

## Examples of graded manifolds.

- (1) Let  $M$  be a manifold. Then the **tangent bundle**  $TM$  is a  $\mathbb{Z}$ -graded manifold with local coordinates:

$$x_j^0 = x_j, \quad \xi_j^1 = dx_j.$$

# Briefly about graded manifolds

Graded manifolds are used for instance in mathematical physics and Poisson geometry.

## Examples of graded manifolds.

- (1) Let  $M$  be a manifold. Then the **tangent bundle**  $TM$  is a  $\mathbb{Z}$ -graded manifold with local coordinates:

$$x_j^0 = x_j, \quad \xi_j^1 = dx_j.$$

- (2) 2-fold (or double) vector bundle:

$$T(T(M)), \quad T(T^*(M)), \dots$$

are  $\mathbb{Z}$ -graded manifolds.

# Graded coverings of super- or graded manifolds

Let us fix the group  $\mathbb{Z}_2$  for **supermanifold**, and  $\mathbb{Z}$  for **graded manifolds**.

Let us fix the following **homomorphism**

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_2, \quad n \longmapsto \bar{n},$$

or more general any homomorphism  $\phi : H_1 \rightarrow H_2$  of finitely generated abelian groups.

For any such a homomorphism we can consider a  $H$ -covering for a supermanifold or graded manifold. Such coverings we call **graded**.

# Graded coverings of supermanifolds

Let  $X$  be a **supermanifold**.

We consider coverings of  $X$  in the category  $\mathcal{C}$  of non-negatively  $\mathbb{Z}$ -graded manifolds corresponding to a homomorphism

$$\phi : \mathbb{Z} \rightarrow \mathbb{Z}_2$$

and satisfying the **universal properties** as in topological case.

- A covering  $p : Y \rightarrow X$ , where  $Y \in \mathcal{C}$ , is **unique** up to isomorphism.
- Let  $X' \in \mathcal{C}$ . Then any morphism  $f : X' \rightarrow X$ , **factors through**  $Y$ :

$$\begin{array}{ccc} & & Y \\ & \nearrow \exists F & \downarrow p \\ X' & \xrightarrow{f} & X \end{array}$$

# Example 1.

## Infinite jet space of a (super)manifold $\mathcal{M}$ .

Coverings of **supermanifolds** in the category of **graded supermanifolds**, corresponding to the following homomorphism

$$\phi : \mathbb{Z} \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2, \quad (m, \bar{n}) \mapsto \bar{n} \in \mathbb{Z}_2$$

of **type**  $\mathbb{Z}^{\geq 0}$ . (For usual manifolds  $\phi : \mathbb{Z}^{\geq 0} \rightarrow 0$  the homomorphism is trivial.)

# Example 1.

## Infinite jet space of a (super)manifold $\mathcal{M}$ .

Coverings of **supermanifolds** in the category of **graded supermanifolds**, corresponding to the following homomorphism

$$\phi : \mathbb{Z} \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2, \quad (m, \bar{n}) \mapsto \bar{n} \in \mathbb{Z}_2$$

of **type**  $\mathbb{Z}^{\geq 0}$ . (For usual manifolds  $\phi : \mathbb{Z}^{\geq 0} \rightarrow 0$  the homomorphism is trivial.)

The **covering projection**  $p : \mathcal{P} \rightarrow \mathcal{M}$  in local charts:

$$p^*(x_i) = x_i + \dot{x}_i + \ddot{x}_i + \dots ;$$

$$p^*(\xi_j) = \xi_j + \dot{\xi}_j + \ddot{\xi}_j + \dots .$$

Here  $x_i$  is even,  $\xi_j$  is odd. The sum is taken over  $\phi^{-1}(\bar{0}) \cap \mathbb{Z}^{\geq 0}$  for  $x_i$  and over  $\phi^{-1}(\bar{1}) \cap \mathbb{Z}^{\geq 0}$  for  $\xi_j$

## Example 2.

Coverings of **supermanifolds** corresponding to the following homomorphism

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_2, \quad \mathbb{Z} \ni n \longmapsto \bar{n} \in \mathbb{Z}_2.$$

of type  $\mathbb{Z}^{\geq 0}$ . (E.V., arXiv:2212.09558)

A covering  $p : \mathcal{P} \rightarrow \mathcal{M}$  is an  $\infty$ -dimensional graded manifold:

$$\mathcal{P} = \varprojlim (\mathcal{P}_2 \leftarrow \mathcal{P}_3 \leftarrow \mathcal{P}_4 \leftarrow \cdots).$$

**A construction of  $\mathcal{P}_2$ :**

“Super Atiyah classes and obstructions to splitting of supermoduli space”, 2014, **Donagi and Witten**.

## Graded covering of $\mathbb{R}^{0|1}$ .

① **Supermanifold:**  $\mathbb{R}^{0|1} = (pt, \wedge(\xi))$ .

② **Graded covering:**

$$\mathcal{P} := (pt, \wedge(\eta_1, \eta_3, \dots))$$

is a graded manifold with  $\deg(\eta_{2k+1}) = 2k + 1$ .

③ **Covering map:**

$$p^*(\xi) = \eta_1 + \eta_3 + \eta_5 + \dots$$

## Universal properties of $\mathcal{P}$ .

- Let  $\mathcal{M} = \mathbb{R}^{0|1}$ ,  $\mathcal{M}'$  be any graded manifold and  $\mathcal{P} = (pt, \wedge(\eta_1, \eta_3, \dots))$ .

$$\begin{array}{ccc}
 & (pt, \wedge(\eta_1, \eta_3, \dots)) & \\
 \nearrow \exists! F & & \downarrow p \\
 \mathcal{M}' & \xrightarrow{f} & \mathbb{R}^{0|1},
 \end{array}$$

$$\begin{array}{ccc}
 & \eta_1 + \eta_3 + \dots & \\
 \nwarrow \exists! F^* & & \uparrow p^* \\
 f^*(\xi) = G_1 + G_3 + \dots & \xleftarrow{f^*} & \xi
 \end{array}$$

- “local diffeomorphism”  $\xi \mapsto \eta_i$ , where  $i = 1, 3, \dots$
- $\mathcal{P}$  is unique up to isomorphism.

## Example (Superquadric $\mathcal{Q}$ )

A superquadric  $\mathcal{Q}$  is a **supermanifold** of dimension  $1|2$  with the base space  $\mathbb{C}\mathbb{P}^1$ , which has the following transition functions

$$y = \frac{1}{x} + \frac{1}{x^3}\xi_1 \cdot \xi_2, \quad \xi_1 \cdot \xi_2 = -\xi_2 \cdot \xi_1;$$
$$\eta_j = \frac{1}{x^2}\xi_j, \quad j = 1, 2.$$

The **graded manifold**  $\mathcal{P}_2$  of degree 2 with the base space  $\mathbb{C}\mathbb{P}^1$  has the following transition functions

$$y = \frac{1}{x}; \quad \text{degree 0}$$
$$\theta_j = \frac{1}{x^2}\zeta_j, \quad j = 1, 2; \quad \text{degree 1}$$
$$w = -\frac{1}{x^2}z + \frac{1}{x^3}\zeta_1 \cdot \zeta_2, \quad \text{degree 2.}$$

## About non-split supermanifolds

The moduli space of super Riemann surfaces is **not split** for  $g \geq 5$ . (Donagi, Witten, 2013).

*“Physically, this (= not split, E.V.) means that certain approaches to superstring perturbation theory that are very powerful in low orders have no close analog in higher orders. Mathematically, it means that the moduli space of super Riemann surfaces cannot be constructed in an elementary way starting with the moduli space of ordinary Riemann surfaces. It has a life of its own.”*

R. Donagi, E. Witten

## Example

The **graded manifold**  $\mathcal{P}_2$  of degree 2 with the base space  $\mathbb{C}\mathbb{P}^1$  has the following transition functions

$$y^0 = \frac{1}{x^0}; \quad \text{degree 0}$$

$$\eta_j^1 = \frac{1}{(x^0)^2} \xi_j^1, \quad j = 1, 2; \quad \text{degree 1}$$

$$y^2 = -\frac{1}{(x^0)^2} x^2 + \frac{1}{(x^0)^3} \xi_1^1 \cdot \xi_2^1, \quad \text{degree 2.}$$

We have the base space  $\mathbb{C}\mathbb{P}^1$ , the locally free sheaf  $\mathcal{E}$  (degree 1), and the following short exact sequence of locally free sheaves

$$0 \rightarrow \wedge^2 \mathcal{E} \rightarrow \mathcal{O}_2 \rightarrow \Omega \rightarrow 0.$$

Manin's Theorem about  $H^1(\Omega^* \otimes \wedge^2 \mathcal{E}) \Leftrightarrow$  the Atiyah Theorem about short exact sequences.

# Graded covering of a Lie superalgebra

Let  $\mathfrak{a}$  be an  $\mathbb{Z}$ -graded Lie superalgebra and

$$\psi : \mathfrak{a} \rightarrow \mathfrak{g}$$

be a homomorphism of Lie superalgebras.

Then there exists unique homomorphism  $\Psi : \mathfrak{a} \rightarrow \mathfrak{h}$  of  $\mathbb{Z}$ -graded Lie superalgebras such that

$$\begin{array}{ccc} & & \mathfrak{h} \\ & \nearrow \exists! \Psi & \downarrow p \\ \mathfrak{a} & \xrightarrow{\psi} & \mathfrak{g} \end{array}$$

Let  $Z \in \mathfrak{a}_s$ , then  $\Psi(Z) = \psi(Z) \otimes t^s$ .

### Example. General linear Lie superalgebra.

$$\mathfrak{gl}_{m|n}(\mathbb{K}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\},$$

The a  $\mathbb{Z}^{\geq 0}$ -covering of  $\mathfrak{g}$  contains all matrices in the following form

$$\begin{pmatrix} A_0 & 0 & 0 & 0 & \cdots \\ C_1 & D_0 & 0 & 0 & \cdots \\ A_2 & B_1 & A_0 & 0 & \cdots \\ C_3 & D_2 & C_1 & D_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

## Example 3.

**Multiplicity free covering of a graded manifold.**

**Covering** of a graded manifold corresponds to the following **homomorphism**

$$\phi : \mathbb{Z} \times \cdots \times \mathbb{Z} \rightarrow \mathbb{Z}, \quad (k_1, \dots, k_n) \mapsto k_1 + \cdots + k_n.$$

- (1) *M. Jotz Lean*. The geometrization of  $N$ -manifolds of degree 2. *Journal of Geometry and Physics*, 133, 113 – 140, 2018.
- (2) *del Carpio-Marek. F.* Geometric structure on degree 2 manifolds. PhD-thesis, IMPA, Rio de Janeiro, 2015.
- (3) *A. Bruce, J. Grabowski and M. Rotkiewicz*, Polarisation of Graded Bundles. *SIGMA* 12 (2016), 106, 30 pages.
- (4) *E. Vishnyakova*. Graded manifolds of type  $\Delta$  and  $n$ -fold vector bundles, *Letters in Mathematical Physics* 109 (2), 2019, 243-293.
- (5) *Miquel Cueva*, Applications of graded manifolds to Poisson geometry, PhD-thesis, IMPA, Rio de Janeiro, 2019.
- (6) *M. Heuer, M.J. Lean*. A geometrisation of  $N$  -manifolds. *Journal de Mathématiques Pures et Appliquées* 184(9), 2024, 64 pages.
- (7) *E. Vishnyakova*. Multiplicity free covering of a graded manifold. *Differential Geometry and its Applications* 2026.

# Applications of graded coverings of supermanifolds

Following a paper in preparation by Luiz F.A. Campos and E.V.

Consider a covering  $\mathcal{P}$  of  $\mathcal{M}$  corresponding to a homomorphism

$$\mathbb{Z} \longrightarrow \mathbb{Z}_2.$$

Recall

$$\mathcal{M} \rightsquigarrow \mathcal{P} = (\mathcal{P}_0, \mathcal{O}), \quad \mathcal{P}_0 = \mathcal{M}_0.$$

We have the following sequence of graded manifolds

$$\mathcal{P}_2 \longleftarrow \mathcal{P}_3 \longleftarrow \mathcal{P}_4 \longleftarrow \mathcal{P}_5 \cdots$$

For any  $n$  the subsheaf (locally free sheaf) of homogeneous elements of weight  $n$  is defined

$$\mathcal{O}_n \subset \mathcal{O}$$

## Example

Let  $n = 2$ . Then the graded manifold  $\mathcal{P}_2$  has the following local graded coordinates

$$(x_i^0, x_j^2), \quad (\xi_s^1, \xi_t^3),$$

where upper index indicates the weight of a coordinate.

The sheaf  $\mathcal{O}_2$  is locally generated by

$$x_i^2, \quad \xi_s^1 \cdot \xi_t^1.$$

We have the following exact sequence

$$0 \rightarrow \wedge^2 \mathcal{E} \rightarrow \mathcal{O}_2 \rightarrow \Omega \rightarrow 0, \quad \mathcal{E} = \mathcal{O}_1.$$

For  $\mathcal{O}_2$  we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_1 \mathcal{O}_1 \longrightarrow \mathcal{O}_2 \longrightarrow \Omega \longrightarrow 0.$$

Here  $\mathcal{O}_1 = \mathcal{E}$ , where  $\text{gr}\mathcal{M} = (\mathcal{M}_0, \wedge \mathcal{E})$ , and  $\Omega$  is the sheaf of 1-form on  $\mathcal{M}_0$ .

Donagi and Witten observed that the **Atiyah class**  $w_2$  of this **sequence** is equal to the first Berezin (Palamodov) class  $b_2$ . It is also called the **first obstruction class**.

Since the  $\mathbb{Z}$ -covering  $\mathcal{P}$  of  $\mathcal{M}$  is a **prolongation** of the Donagi-Witten construction, ( $[V.]$ ), we also can look at the further Atiyah classes.

For example, for  $\mathcal{O}_3$  we have two exact sequences

$$0 \longrightarrow \bigwedge^3 \mathcal{O}_1 \longrightarrow \mathcal{O}_3 \longrightarrow \mathcal{O}_3 / \bigwedge^3 \mathcal{O}_1 \longrightarrow 0,$$

we denote the corresponding **Atiyah class** by  $w_3$ , and

$$0 \longrightarrow \mathcal{O}_1 \mathcal{O}_2 \longrightarrow \mathcal{O}_3 \longrightarrow \mathcal{E} \longrightarrow 0.$$

We denote the corresponding **Atiyah class** by  $v_3$ .

Considering  $\mathcal{O}_n$ , we will get two series of classes

$$(w_2, w_3, w_4, \dots), \quad (v_2, v_3, v_4, \dots),$$

where  $w_2 = v_2 = b_2$  is the first obstruction class.

## About Berezin (Palamodov) classes.

Let  $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$  be a supermanifold,  $\mathcal{J}$  be an ideal generated by odd elements.

Denote

$$\mathcal{M}_{(n)} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}}/\mathcal{J}^{n+1}).$$

We put

$$G_i = \{\alpha \in \text{Aut}(\text{gr}\mathcal{O}_{\mathcal{M}}) \mid \alpha = id \pmod{\mathcal{J}^i}\}$$

We have the following exact sequences of sheaves of groups

$$e \longrightarrow G_{i+1} \longrightarrow G_i \longrightarrow T_{(-1)^i} \otimes \bigwedge^i \mathcal{E} \longrightarrow 0.$$

The class  $\gamma \in H^1(G_2)$  corresponding to  $\mathcal{M}$  we call the Green class of  $\mathcal{M}$ .

The first Berezin (Palamodov) class  $b_2$  is the image of the Green class  $\gamma$  by the following map

$$H^1(G_2) \longrightarrow H^1(\Omega^* \otimes \bigwedge^2 \mathcal{E}).$$

Recall that the class  $w_2 = v_2 = b_2$  is also the Atiyah class of the exact sequence

$$0 \longrightarrow \bigwedge^2 \mathcal{E} \longrightarrow \mathcal{O}_2 \longrightarrow \Omega \longrightarrow 0.$$

Hence  $w_2 \in H^1(\Omega^* \otimes \bigwedge^2 \mathcal{E})$ .

We also have

$$e \longrightarrow G_4 \longrightarrow G_2 \longrightarrow \mathcal{T}_2 \longrightarrow 0.$$

### Theorem (—, Campus)

*The image of the Green class  $\gamma$  of the supermanifold  $\mathcal{M}$  by the following map*

$$H^1(G_2) \longrightarrow H^1(\mathcal{T}_2).$$

*is equal to  $w_3$ .*

## More about Berezin (Palamodov) classes

- (1) The class  $b_2$  is always defined, but the class  $b_3$  is defined if  $b_2 = 0$ . And so on.
- (2) If  $b_2 \neq 0$ , then  $\mathcal{M}$  is not split.
- (3) If  $b_2 = 0, \dots, b_m = 0$ , then the supermanifold  $\mathcal{M}$  is split.
- (4) If  $b_2 = 0$ , hence  $b_3$  is defined. It may happen that  $b_3 \neq 0$ , but the supermanifold is split.

**Classes  $(w_2, w_3, \dots, w_m)$  are defined for any supermanifold.**

**Theorem (Luiz F.A. Campos, E.V.)**

*If  $\mathcal{M}$  is split, then  $w_2 = 0, \dots, w_m = 0$ .*

Further,

**Theorem (Luiz F.A. Campos, E.V.)**

*If the Berezin (Palamodov) class*

$$b_k \in H^1(T_{(-1)^k} \otimes \bigwedge \mathcal{E})$$

*is defined, then  $w_k$  is the image of  $b_k$  under a natural map.*

Hence it may happen that  $b_k \neq 0$ , the supermanifold  $\mathcal{M}$  is split, but we have  $w_k = 0$ .

**Classes  $(v_2, v_3, \dots, v_m)$  are defined for any supermanifold.**

Let  $\mathcal{M}$  be any supermanifold,  $\text{gr}\mathcal{M}$  be the corresponding split supermanifold. Denote by  $\widehat{v}_k$  the classes of  $\text{gr}\mathcal{M}$ . We have  $\widehat{v}_2 = 0$ .

**Theorem (Luiz F.A. Campos, E.V.)**

*$\mathcal{M}$  is split if and only if  $(v_2, \dots, v_m) \sim (\widehat{v}_2, \dots, \widehat{v}_m)$ .*

**Classes  $(v_2, v_3, \dots, v_m)$  are defined for any supermanifold.**

Let  $\mathcal{M}$  be any supermanifold,  $\text{gr}\mathcal{M}$  be the corresponding split supermanifold. Denote by  $\widehat{v}_k$  the classes of  $\text{gr}\mathcal{M}$ . We have  $\widehat{v}_2 = 0$ .

**Theorem (Luiz F.A. Campos, E.V.)**

*$\mathcal{M}$  is split if and only if  $(v_2, \dots, v_m) \sim (\widehat{v}_2, \dots, \widehat{v}_m)$ .*

As a corollary we get Koszul's Theorem:

**Corollary (Luiz F.A. Campos, E.V.)**

*If  $\mathcal{M}$  admits an affine connection, then we have*

$$v_k = 0, \quad \widehat{v}_k = 0, \quad \forall k.$$

*In particular,  $v \sim \widehat{v}$  (Koszul's Theorem).*

## Berezin (Palamodov) retraction of a supermanifold.

Let  $\mathcal{M}$  be a supermanifold with  $b_2 = 0$ . Then in Berezin's book, Theorem 4.6.2 suggests an explicit change of coordinates such that, after this transformation, we have modulo  $\mathcal{J}^3$ :

$$y_i = F_i(x), \quad \eta_j = G(x)_j^a \xi_a + G(x)_j^{stq} \xi_s \xi_t \xi_q.$$

Further, if  $b_3 = 0$  we can continue this process.

**What to do if  $b_3 \neq 0$ ?**

**New retraction method.** [Luiz F.A. Campos, E.V.]

Theorem (Luiz F.A. Campos, E.V.)

$$\mathcal{M} \simeq \text{gr}\mathcal{M} \quad \text{if and only if} \quad \mathcal{P}_{\mathcal{M}} \simeq \mathcal{P}_{\text{gr}\mathcal{M}}.$$

**New retraction method.** [Luiz F.A. Campos, E.V.]

Theorem (Luiz F.A. Campos, E.V.)

$$\mathcal{M} \simeq \text{gr}\mathcal{M} \quad \text{if and only if} \quad \mathcal{P}_{\mathcal{M}} \simeq \mathcal{P}_{\text{gr}\mathcal{M}}.$$

### Idea of our retraction method

Let  $\mathcal{O}$  be the structure sheaf of  $\mathcal{P}$  and  $\widehat{\mathcal{O}}$  be the structure sheaf of  $\mathcal{P}_{\text{gr}\mathcal{M}}$ . Then we have two possibilities

- 1  $v_2 = b_2 = w_2 \neq 0$ , hence the supermanifold is not split.
- 2  $v_2 = b_2 = w_2 = 0$ , we apply the following procedure.

If  $v_2 = 0$ , then there exists a bundle isomorphism such that the following diagram is commutative

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_1 & \longrightarrow & \mathcal{O}_2 & \longrightarrow & \Omega \longrightarrow 0 \\
 & & \downarrow \text{id} & & \downarrow \Phi'_2 & & \downarrow \text{id} \\
 0 & \longrightarrow & \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_1 & \longrightarrow & \widehat{\mathcal{O}}_2 & \longrightarrow & \Omega \longrightarrow 0
 \end{array}$$

If  $v_2 = 0$ , then there exists a bundle isomorphism such that the following diagram is commutative

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_1 & \longrightarrow & \mathcal{O}_2 & \longrightarrow & \Omega \longrightarrow 0 \\
 & & \downarrow \text{id} & & \downarrow \Phi'_2 & & \downarrow \text{id} \\
 0 & \longrightarrow & \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_1 & \longrightarrow & \widehat{\mathcal{O}}_2 & \longrightarrow & \Omega \longrightarrow 0
 \end{array}$$

Such an isomorphism  $\Phi'_2$  is not unique, it is defined up to automorphism of  $\widehat{\mathcal{O}}_2$ , which is identical on  $\widehat{\mathcal{O}}_1 = \mathcal{E}$  and  $\Omega$ .

We apply the graded change of coordinates in  $\mathcal{P}$ , induced by  $\Phi'_2$ . (This step is close to the Berezin method, but in the sheaf  $\mathcal{O}_2$ .)

Consider the exact sequence for the class  $v_3$  in our new coordinates:

$$0 \longrightarrow \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_2 \longrightarrow \mathcal{O}_3 \longrightarrow \mathcal{E} \longrightarrow 0.$$

We see that

$$v_3, \widehat{v}_3 \in H^1(\mathcal{E}^* \otimes \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_2).$$

So we can compare these classes.

Consider the exact sequence for the class  $v_3$  in our new coordinates:

$$0 \longrightarrow \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_2 \longrightarrow \mathcal{O}_3 \longrightarrow \mathcal{E} \longrightarrow 0.$$

We see that

$$v_3, \widehat{v}_3 \in H^1(\mathcal{E}^* \otimes \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_2).$$

So we can compare these classes.

Denote by  $Aut_2$  the group of **graded** automorphisms of the sheaf  $\widehat{\mathcal{O}}_2$ , which are identical on  $\mathcal{O}_1$  and  $\Omega$ . We have two possibilities

- 1  $v_3 \notin Aut_2 \cdot \widehat{v}_3$ , hence the supermanifold is not split.
- 2  $v_3 \in Aut_2 \cdot \widehat{v}_3$ , then we have  $v_3 = \alpha^{-1} \cdot \widehat{v}_3$  for some  $\alpha^{-1} \in Aut_2$ .

In the case  $v_3 = \alpha^{-1} \cdot \widehat{v}_3$ , we apply coordinate change  $\alpha$  to  $\mathcal{O}$ , and continue in a similar way.

In other words, in this case we can find bundle isomorphism  $\Phi_2$  such that the following diagram is commutative

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathcal{O}_1 \mathcal{O}_1 & \longrightarrow & \mathcal{O}_2 & \longrightarrow & \Omega & \longrightarrow & 0 \\
 & & \downarrow \text{id} & & \downarrow \Phi_2 & & \downarrow \text{id} & & \\
 0 & \longrightarrow & \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_1 & \longrightarrow & \widehat{\mathcal{O}}_2 & \longrightarrow & \Omega & \longrightarrow & 0
 \end{array}$$

and in addition  $\Phi_2(v_3) = \widehat{v}_3$ .

And so on. Since we have  $\Phi_2(v_3) = \widehat{v}_3$ , there exists a bundle isomorphism  $\Phi_3$ :

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathcal{O}_1 \mathcal{O}_2 & \longrightarrow & \mathcal{O}_3 & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow \Phi_2 & & \downarrow \Phi'_3 & & \downarrow \text{id} & & \\
 0 & \longrightarrow & \widehat{\mathcal{O}}_1 \widehat{\mathcal{O}}_2 & \longrightarrow & \widehat{\mathcal{O}}_3 & \longrightarrow & \Omega & \longrightarrow & 0
 \end{array}$$

We repeat the procedure for  $v_4$  and  $\widehat{v}_4$ :

- 1  $v_4 \notin \text{Aut}_3 \cdot \widehat{v}_4$ , hence the supermanifold is not split.
- 2  $v_4 \in \text{Aut}_3 \cdot \widehat{v}_4$ , then we have  $v_4 = \alpha^{-1} \cdot \widehat{v}_4$  for some  $\alpha^{-1} \in \text{Aut}_3$ . Hence, we can construct the next isomorphism  $\Phi'_4 : \mathcal{O}_4 \longrightarrow \widehat{\mathcal{O}}_4$ .

### Conclusion.

Berezin method works if all classes  $b_k = 0$ .

Our method always gives an answer, whether a supermanifold is split or not.