Tracking the symmetries of $\mathbb{Z}_3$-orbifold K3s within the Mathieu groups

Ida Zadeh

University of Southampton

Institute of Mathematics

Czech Academy of Sciences

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with K. Budzik, A. Taormina, M. Ungureanu, K. Wendland

Kasia Budzik

Anne Taormina

Mara Ungureanu

Katrin Wendland

Women in Mathematical Physics collaboration

Goal

Find all geometric symmetries of $\mathbb{Z}_3$-orbifold K3 surfaces $X = \widetilde{T/\mathbb{Z}_3}$. Track them in sporadic groups M_{12} and M_{24}.

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How ?

1. Construct X via $\mathbb{Z}_3$-orbifolding T and blowing up the singularities.
2. Give the full description of the integral cohomology lattice $H^2(X, \mathbb{Z})$ in terms of the 'Kummer-like' lattice (P) and its orthogonal complement (K) in $H^2(X, \mathbb{Z})$ using lattice gluing [Nikulin]. P is a minimal primitive sublattice which contains contributions from blow-ups.
3. Determine the group of symmetries of X and its action on $H^2(X, \mathbb{Z})$, P, and K.
4. Map P into a Niemeier lattice and embed it primitively.
5. Track symmetry group of X in M_{12} and M_{24} (automorphisms of Niemeier lattices) explicitly in terms of permutations on 12, resp. 24 elements.

Results

Proposition 1

The symmetry group of X is $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4$ or $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$, depending on symmetries of the underlying torus T. Here $(\mathbb{Z}_3)^2$ is the group of translational symmetries, induced by translations on T, and $\mathbb{Z}_4$ or $\mathbb{Z}_2$ is the group of rotational symmetries, induced by action of a subgroup of $SU(2)$ on T.

Proposition 2

Let ϑ denote the group homomorphism that maps the symmetry group of X into M_{24}. Then the image of the symmetry group of X in M_{24} is generated by

$$\begin{aligned}
\vartheta(\alpha^1) &= (1,3,8)(2,23,16)(5,22,14)(10,20,21)(11,17,24)(12,13,18)\,, \\
\vartheta(\alpha^2) &= (1,23,12)(2,18,8)(3,16,13)(5,14,22)(7,15,19)(10,20,21)\,, \\
\vartheta(\beta) &= (1,8,18,12)(2,13,23,3)(5,10)(7,11,15,24)(14,21,22,20)(17,19)\,.
\end{aligned}$$

Results

Theorem

The subgroup of type $(\mathbb{Z}_2)^4 \rtimes A_8$ *of* M_{24} *which is obtained as combined symmetry group of all Kummer K3s* $(T/\mathbb{Z}_2)$ *[Taormina, Wendland], together with the image of the symmetry group of* $\mathbb{Z}_3$*-orbifold K3 surfaces under the map* ϑ *generate the Mathieu group* M_{24}*.*

Main tool: Lattice theory

The main idea is to use lattice theoretic techniques to relate geometry and symmetries.

Symmetries of K3 surfaces are related to symmetries of $+$-definite even self-dual lattices of rank 24 (Niemeier lattices). The latter are well known and have been classified. This provides a tool to study symmetries of K3 non-linear sigma-models.

The connection is realised via the *gluing technique*: how to construct an even self dual lattice via adjoining two even lattices. [Nikulin]

- Motivation: Mathieu Moonshine

- $\mathbb{Z}_3$-orbifold K3s and lattices

- Representation of symmetries of $\mathbb{Z}3$-orbifold K3s on a Niemeier lattice

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K3 surface

A K3 surface X is a simply connected compact complex surface with a trivial canonical bundle. There exists a holomorphic 2-form on X which vanishes nowhere.

Symplectic automorphisms: leave the nowhere-vanishing holomorphic 2-form and some Kähler class ω invariant.

Any finite group of symplectic automorphisms of a K3 surface is isomorphic to a subgroup of M_{23}. [Mukai; Kondo]

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The integral cohomology group $H^2(X, \mathbb{Z})$ of a K3 surface, endowed with a bilinear form, is an even self dual lattice of signature $(3, 19)$

$$H^2(X, \mathbb{Z}) \simeq U^3 \oplus E_8^2(-1) \ .$$ [Milnor]

U is the hyperbolic lattice with signature $(1, 1)$ and quadratic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Elliptic genus of K3

The elliptic genus of K3 is a weak Jacobi form of weight 0 and index 1 [Eguchi, Ooguri, Taormina, Yang]

$$\mathcal{E}_{\mathrm{K3}} = 8\left[\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + \left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + \left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2\right]$$

$$q := e^{2\pi i\tau}\ , \quad y := e^{2\pi i z}\ , \qquad \tau \in \mathbb{H}\ , \quad z \in \mathbb{C}$$

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \eta e^{2\pi i\alpha\beta} q^{\frac{\alpha^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^{n+\alpha-\frac{1}{2}}e^{2\pi i\beta})(1+q^{n-\alpha-\frac{1}{2}}e^{-2\pi i\beta})$$

$$\vartheta_1 = \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}, \vartheta_2 = \vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}, \vartheta_3 = \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}, \vartheta_4 = \vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}, \qquad \eta = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\ .$$

The elliptic genus is a topological quantity.

$$\mathcal{E}_{\mathrm{K3}} = 2y + 20 + \frac{2}{y} + q\left(20y^2 - 128y + 216 - \frac{128}{y} + \frac{20}{y^2}\right) + O(q^2)$$

Mathieu moonshine

Dimensions of representations of Mathieu group M_{24} appear in the decomposition of $\mathcal{E}_{\mathrm{K3}}$ into small $\mathcal{N} = 4$ superconformal characters [Eguchi, Ooguri, Tachikawa]

$$\mathcal{E}_{\mathrm{K3}} = 20\mathrm{ch}_{\frac{1}{4},0} - 2\mathrm{ch}_{\frac{1}{4},\frac{1}{2}} + \mathbf{90}\mathrm{ch}_{\frac{1}{4}+1,0} + \mathbf{462}\mathrm{ch}_{\frac{1}{4}+2,0} + \mathbf{1540}\mathrm{ch}_{\frac{1}{4}+3,0} + \cdots ,$$

$$\mathrm{ch}^{\mathcal{N}=4}_{h,\ell=0} := q^{h-\frac{3}{8}} \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^3} .$$

[Eguchi, Taormina]

$$\begin{aligned} \mathbf{90} &= \mathbf{45} + \overline{\mathbf{45}} \\ \mathbf{462} &= \mathbf{231} + \overline{\mathbf{231}} \\ \mathbf{1540} &= \mathbf{770} + \overline{\mathbf{770}} \end{aligned}$$

Decomposition of the elliptic genus in terms of M_{24} representations is understood. [Gannon]

Mathieu moonshine - cont'd

The vertex algebra underlying the M_{24} group is still unknown!

Elliptic genus arises in K3 non-linear sigma-models: What can we learn about Mathieu moonshine from K3 sigma-models?

K3 sigma-models: 2d CFTs defined on a Riemann surface and with target space a K3 surface. Sigma-model on K3 has an 80-d moduli space

$$\mathcal{M}_{K3} = \mathrm{O}(4,20;\mathbb{Z})\backslash \mathrm{O}(4,20;\mathbb{R})/\mathrm{O}(4;\mathbb{R}) \times \mathrm{O}(20;\mathbb{R}) \ .$$

[Aspinwall, Morrison; Wendland; Nahm, Wendland]

Mathieu moonshine - cont'd

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[Aspinwall, Morrison; Wendland; Nahm, Wendland]

There is however *no* K3 sigma-model with M_{24} as its automorphism group. Many of them have symmetries which are not even included in M_{24} ! [Gaberdiel,Hohenegger,Volpato]

How to pin down the action of M_{24} ?
Geometric: Symmetry surfing, Non-geometric: String theory.

Symmetry surfing programme is built on the idea that the Mathieu Moonshine phenomenon arises through symplectic automorphisms of K3 surfaces *only*. Combine geometric symmetries from distinct points of $\mathcal{M}_{\mathrm{K3}}$ moduli space. [Taormina, Wendland]

Extensive work done on Kummer surfaces, i.e. $\mathbb{Z}_2$-orbifold K3s. Symmetry surfing the moduli space of Kummer surfaces allows one to naturally combine symmetry groups from distinct points in moduli space through a common polarization and obtain the octad subgroup $(\mathbb{Z}_2)^4 \rtimes A_8$ of M_{24}, which is a maximal subgroup.

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The goal of this work is to realise the group of symplectic automorphisms of $\mathbb{Z}_3$-orbifold K3s as a subgroup of M_{12} and M_{24}.

Combining the generators of the two symmetry groups generates M_{24}. Realisation of symmetry group of Z3-orbifold K3s as a subgroup of M_{24} currently relies on ad hoc choices, and requires a geometric or conformal field theoretic justification for how these symmetry groups can be combined via symmetry-surfing.

- Motivation: Mathieu Moonshine

- $\mathbb{Z}_3$-orbifold K3s and lattices

- Representation of symmetries of $\mathbb{Z}3$-orbifold K3s on a Niemeier lattice

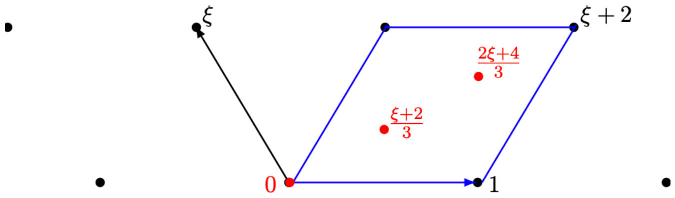
$\widetilde{T/\mathbb{Z}_3}$

T is a complex 2-torus given as a product of two elliptic curves with $\mathbb{Z}_3$ symmetry, $T = \mathbb{C}^2/L$, where the lattice L is generated by

$$\lambda_1 = (1,0),\ \ \lambda_2 = (\xi, 0),\ \ \lambda_3 = (0,1),\ \ \lambda_4 = (0,\xi); \qquad \xi = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i\ .$$

$\mathbb{Z}_3$-action is generated by $(z_1, z_2) \mapsto (\xi z_1, \xi^{-1} z_2)$.

$(z_1, z_2) \in T$ is a fixed point if and only if $(\xi z_1, \xi^{-1} z_2) - (z_1, z_2) \in L$. There are 9 fixed points given by $(t_1, t_2)\frac{\xi+2}{3}$, $(t_1, t_2) \in \mathbb{F}_3^2$, where $\mathbb{F}_3$ denotes the field with three elements $0, 1, -1$.

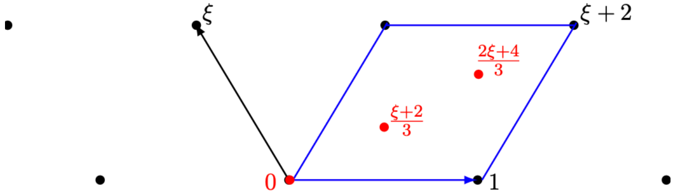


$\widetilde{T/\mathbb{Z}_3}$

$\mathbb{Z}_3$-orbifold is obtained by taking the quotient $T/\mathbb{Z}_3$ and minimally resolving each of the 9 singularities.

Blowing up once resolves singularitues. Exceptional divisors of the blow up decompose into two irreducible components, each isomorphic to $\mathbb{P}^1$, intersecting at one point. These are singularities of type A_2.

Minimally resolved The $\mathbb{Z}_3$-orbifold $X = \widetilde{T/\mathbb{Z}_3}$ is a K3 surface.



Integral cohomology

K3 lattice: $H^2(X,\mathbb{Z}) \simeq U^3 \oplus E_8^2(-1)$.

Symmetry group of $X = \widetilde{T/\mathbb{Z}_3}$ is the group of automorphisms preserving holomorphic volume form and Kähler class induced by those of underlying T. Our goal is to give the action of symmetry group of X on $H^2(X,\mathbb{Z})$.

To this end, we consider two sublattices of $H^2(X,\mathbb{Z})$:

1. Smallest primitive $P \subset H^2(X,\mathbb{Z})$ with sign. $(0,18)$. Contains Poincaré duals of exceptional divisors resulting from blowing up singularities of $T/\mathbb{Z}_3$ (*Kummer-like* lattice).

2. Smallest primitive $K := P \cap H^2(X,\mathbb{Z})$ with sign. $(3,1)$. Contains Poincaré duals of torus cycles in general position (i.e. do not contain fixed points).

Let us first review some facts in lattice theory.

Lattice

A set of points in a N-dim vector space V:

$$L = \Big\{ \sum_{i=1}^{N} n_i e_i \Big| n_i \in \mathbb{Z} \Big\} \ ,$$

with basis vectors e_i and symmetric bilinear form $\langle \ , \ \rangle$. V may be $\mathbb{R}^N$ or $\mathbb{R}^{p,q}$, $p + q = N$.

The *unit cell* $\{x = \sum_i x_i e_i | 0 \leq x_i < 1\}$ contains one lattice point $(x = 0)$ and

$$\mathrm{vol}(L) = \sqrt{|\det G|} \ , \ G_{ij} := \langle e_i, e_j \rangle \ .$$

Dual lattice $L^* := \{w \in V | \langle w, v \rangle \in \mathbb{Z} \ , \forall v \in L\}$, $\mathrm{vol}(L) = (\mathrm{vol}(L^*))^{-1}$.

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Dual lattice $L^* := \{w \in V | \langle w, v \rangle \in \mathbb{Z} \, , \forall v \in L\}$, $\text{vol}(L) = (\text{vol}(L^*))^{-1}$.

Lattice L is

- *integral* if $\langle v, w \rangle \in \mathbb{Z}$, $\forall v, w \in L$ (iff $L \subseteq L^*$),
- *even* if the $\langle x, x \rangle = 0 \mod 2$, $\forall x \in L$,
- *self-dual* if $L = L^*$ (hence $\text{vol}(L) = \text{vol}(L^*) = 1$, $|\det G| = 1$).

If L is integral, $A_L := L^*/L$ is an abelian group of order $|\det G|$, called the *discriminant group*. E.g. $A_1^*/A_1 = \mathbb{Z}_2$.

Lattice – cont'd

Orthogonal complement of $I \subset L$ is $I^\perp := \{y \in L | \langle y, x\rangle = 0 \ , \forall x \in I\}$.

Discriminant quadratic form of an even lattice L is the extension of bilinear form on L to the one on L^* and takes values in $\mathbb{Q}$

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z} \ , \qquad q_L(x + L) = \langle x, x\rangle + 2\mathbb{Z} \ , x \in L^* \ .$$

Primitive sublattice $I \subset L$ if L/I is a free abelian group: $I = (I \otimes \mathbb{Q}) \cap L$.

Lattice – cont'd

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Primitive sublattice $I \subset L$ if L/I is a free abelian group: $I = (I \otimes \mathbb{Q}) \cap L$.

Lattice gluing

[Nikulin]

Suppose L is an even self-dual lattice. If $I \subset L$ is primitive, then there is an isomorphism $\iota : A_I \xrightarrow{\cong} A_{I^\perp}$ such that $q_{I^\perp}(\iota(a)) = -q_I(a)$ for $a \in A_I$.

Conversely, given two even lattices L_1 and L_2 with $\iota : A_{L_1} \xrightarrow{\cong} A_{L_2}$ and $q_{L_2} \circ \iota = -q_{L_1}$, one can adjoin cosets to construct an even self-dual lattice

$$\Gamma = \{x \oplus y \in L_1^* \oplus L_2^* | \; \iota([x]) = [y]\} \;,$$

where $[x]$ and $[y]$ are images of $x \in L_1^*$ and $y \in L_2^*$ in A_{L_1} and A_{L_2} resp. $x \oplus y$ are the glue vectors.

Ex. $L_1 = A_1$, $L_2 = A_1(-1)$

$A_{A_1} = \mathbb{Z}_2$ with conjugacy classes $\{[0]_+, [f]_+\}$

$A_{A_1(-1)} = \mathbb{Z}_2$ with conjugacy classes $\{[0]_-, [f]_-\}$

$U = \{x \oplus y \in A_1^* \oplus A_1^*(-1) |\ \iota([f]_+) = [f]_-\}$

Ex. $L_1 = A_1$, $L_2 = E_7$, $\quad A_{A_1} \cong A_{E_7} \cong \mathbb{Z}_2$

$E_8 = \{x \oplus y \in A_1^* \oplus E_7^* |\ \iota([\mathrm{w}_2]) = [\mathrm{w}_{56}]\}$

Ex. Glue 24 D_1 root lattices to construct the Leech lattice !

K3 lattice. Glue $P \subset H^2(X, \mathbb{Z})$ and $K = P^\perp \cap H^2(X, \mathbb{Z}) \cong U(3) \oplus \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ to construct $H^2(X, \mathbb{Z})$

$$A_P \cong A_K \cong \mathbb{Z}_3^3 , \quad \iota : A_K \xrightarrow{\cong} A_P , \quad \text{and} \quad q_P \circ \iota = -q_K$$

Symmetries of $\mathbb{Z}_3$-orbifold K3s

Symmetry group of T which descends to $T/\mathbb{Z}_3$ is $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4$ where $(\mathbb{Z}_3)^2$ acts by translations (generated by α_1 and α_2) and $\mathbb{Z}_4$ acts by rotations (generated by β).

The entire symmetry group of (X, ω) is induced by the symmetries of the underlying torus.

Proposition

The symmetry group of (X, ω) is $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4$, where $(\mathbb{Z}_3)^2$ is the group of translational symmetries, induced by translations on the underlying torus T, and $\mathbb{Z}_4$ is the group of rotational symmetries, induced by the action of a subgroup of $SU(2)$ on the underlying torus T.

This determines the action of symmetry group of (X, ω) on $H^2(X, \mathbb{Z})$, P, and K. Symmetries correspond to a group of lattice automorphisms that map K to K and P to P.

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Embedding in Niemeier lattices

Lattice P contains as a proper sublattice the rank-18 root lattice R of type A_2^9 generated by components of the exceptional divisors for blow ups of the singularities of $T/\mathbb{Z}_3$.

In order to track symmetries of X, it is natural to view them as permutation groups, acting on each set of vectors of equal length in lattice $H^2(X, \mathbb{Z})$. However, since this lattice has signature $(3, 19)$, every non-zero lattice vector is a member of an infinite set of lattice vectors of the same length. It is therefore more convenient to represent symmetries on a lattice of definite signature.

Since each symmetry of X is uniquely determined by its action on P, we search for primitive embeddings of the lattice $P(-1)$ in some Niemeier lattice.

A Niemeier lattice is an even self-dual positive definite lattice of rank 24. There are 24 isomorphism classes of Niemeier lattices uniquely determined by their root lattice, i.e. by the sublattice generated by all lattice vectors of length squared 2. [Niemeier]

Embedding of $P(-1)$ in Niemeier lattices

Proposition

The primitive embedding of $P(-1)$ in the Niemeier lattice N is unique up to automorphisms of N.

Proposition

Assume that $\widehat{N}$ is a Niemeier lattice which allows a primitive embedding of $P(-1)$. Then $\widehat{N}$ is a Niemeier lattice of type A_2^{12}.

There exists one *non-primitive* embedding of $P(-1)$ into a Niemeier lattice of different type, which can only be of type E_6^4.

Tracking symmetries of $\mathbb{Z}_3$-orbifold K3s X in M_{12}

Primitive embedding $\gamma : P(-1) \hookrightarrow N$ has image $\widetilde{P}$. Since $\widetilde{P}$ is a primitive sublattice of N, one may use Nikulin's gluing techniques to describe N in terms of $\widetilde{P}$ and $\widetilde{K} := \widetilde{P} \cap N$. This allows us to represent symmetry group of X on $\widetilde{P}$ and $\widetilde{K}$.

Proposition

The symmetry group of (X, ω) acts faithfully on the Niemeier lattice N. The resulting subgroup of $\mathrm{Aut}(N)$, which is isomorphic to $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4$, can be generated by the automorphisms $\widetilde{\alpha}^1$, $\widetilde{\alpha}^2$, and $\widetilde{\beta}$ induced by translational and rotational symmetries α^1, α^2, and β.

This leads to an injective group homomorphism Θ mapping the symmetry group of X into the Mathieu group M_{12}. The image is generated by the following three permutations:

$$\Theta(\alpha^1) = (1,4,3)(2,9,7)(5,8,6)\,, \qquad \Theta(\alpha^2) = (1,9,8)(2,5,3)(4,7,6)\,,$$
$$\Theta(\beta) = (1,9,2,3)(4,8,7,5)\,.$$

Tracking symmetries of $\mathbb{Z}_3$-orbifold K3s X in M_{24}

Since $M_{12} \subset M_{24}$, we can track symmetries of $\mathbb{Z}_3$-orbifold K3s within M_{24}.

Proposition

Let ϑ denote the group homomorphism that maps the symmetry group of X into M_{24}. Then the image of the symmetry group of X in M_{24} is generated by

$$\begin{aligned}\vartheta(\alpha^1) &= (1,3,8)(2,23,16)(5,22,14)(10,20,21)(11,17,24)(12,13,18)\,,\\ \vartheta(\alpha^2) &= (1,23,12)(2,18,8)(3,16,13)(5,14,22)(7,15,19)(10,20,21)\,,\\ \vartheta(\beta) &= (1,8,18,12)(2,13,23,3)(5,10)(7,11,15,24)(14,21,22,20)(17,19)\,.\end{aligned}$$

Theorem

The subgroup of type $(\mathbb{Z}_2)^4 \rtimes A_8$ of M_{24} which is obtained as combined symmetry group of all Kummer K3s $(T/\mathbb{Z}_2)$ [Taormina, Wendland], together with the image of the symmetry group of $\mathbb{Z}_3$-orbifold K3 surfaces under the map ϑ generate the Mathieu group M_{24}.

Thank You !

Even self-dual lattice $\Gamma_{(\gamma_+,\gamma_-)}$

$\Gamma_{(\gamma_+,\gamma_-)}$ exists iff $\gamma_+ - \gamma_- = 0$ mod 8.

Indefinite (Lorentzian) $\Gamma_{(\gamma_+,\gamma_-)}$, $\gamma_+ > 0$, $\gamma_- > 0$ is unique upto lattice isometries [Milnor]

$$\Gamma_{(\gamma_+,\gamma_-)} \cong E_8^{\frac{\gamma_+ - \gamma_-}{8}} \oplus U^{\min(\gamma_+,\gamma_-)} \; , \quad G_U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \; .$$

Leech lattice Λ

Unique $+$-def. even self dual lattice of rank 24 without roots.

See Borcherds' video on YouTube here ! (In dim. $32 > 10^7$ examples !)

$O(\Lambda) = \mathrm{Co}_0$. There are 290 classes of sublattices of Λ fixed (up to conjugacy) by elements of Co_0. Identified by $\tilde{I}$ $(24-s, 0)$ and N $(s, 0)$, $\tilde{I}^*/\tilde{I} \cong N^*/N$. [Höhn, Mason]