CFT correlation determinants with elliptic functions

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Introduction

Computations [Kac, Zhu, Dong-Li-Mason, Mason-Tuite-Z, Tuite-Z] related to vertex algebras were always a source of new identities in number theory. While finding closed expressions for vertex operator algebra characters, we finally obtain relations among modular forms, fundamental kernels, and q-series.

Various ways to compute characters for vertex operator algebra modules lead to generation of modular forms as well as interesting identities for them in terms of elliptic functions. In particular, [Mason-Tuite-Z] considerations of the twisted partition function on the torus for the rank two free fermion vertex operator superalgebra allow us to provide a pure algebraic explanation of Jacobi triple product identity [Kac].

At the same time, computations of higher correlation functions on a genus one Riemann surface provide us [Mason-Tuite-Z] with an elliptic version of the Fay's trisecant identity [Fay] known from algebraic geometry. Various identities for powers of the η-function appear [Kac] in studies of affine Lie algebras. Those are important in number theory.

Torus correlation functions

For an automorphism g twisted module $\mathcal{V}$ for a vertex operator algebra V we find closed formulas for correlation functions of vertex operators $\mathcal{Y}$ on the torus, $q = e^{2\pi i \tau}$, with local coordinates z_i , $v_i \in V$, $1 \leqslant i \leqslant n$, [Mason-Tuite-Z]:

$$Z_V^{(1)} \left[\begin{array}{c} f \\ g \end{array} \right] (v_1, z_1, \ldots, v_n, z_n; q) = \mathrm{Tr}_{\mathcal{V}} \left(f \mathcal{Y}(v_1, z_1) \ldots \mathcal{Y}(v_n, z_n) \; q^{L(0) - C/24} \right),$$

where $L(0)$ is the Virasoro algebra generator, and C is central charge.

The formal parameter is associated to a complex parameter on the torus. Final expressions are given by determinants of matrices with elements being coefficients in the expansions of the regular parts of corresponding differentials: Bergman (bosons) or Szegő (fermions) kernels [Mason-Tuite-Z].

In this talk we derive some new genus two generalizations of the fundamental formulas for powers of the η-function in terms of deformed versions [Dong-Li-Mason, Mason-Tuite-Z] of Weierstrass functions and Eisenstein series.

In particular, we find that powers of the modular discriminant are expressed (up to theta-functions multipliers) via determinants of finite matrices containing combinations of deformed modular functions. In the proof we use the generalized elliptic version of the Fay's trisecant identity for a vertex operator superalgebra.

Modular discriminant and Eisenstein series

The modular discriminant is defined by $\Delta(\tau) = \eta(\tau)^{24}$, where $\eta(\tau)$ is the Dedekind eta-function, $q = e^{2\pi i\tau}$, $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$.
Recall $E_n(\tau)$ is equal to 0 for n odd, and for n even is the Eisenstein series [Serre]

$$E_n(\tau) = -\frac{B_n(0)}{n!} + \frac{2}{(n-1)!}\sum_{r\geqslant 1}\frac{r^{n-1}q^r}{1-q^r},$$

where $B_n(0)$ is the n-th Bernoulli number

$$\frac{q_z^{\lambda}}{q_z-1} = \frac{1}{z} + \sum_{n\geqslant 1}\frac{B_n(\lambda)}{n!}z^{n-1}.$$

One finds [Ramanujan] the relations: $E_8 = E_4^2$, $E_{10} = E_4E_6$, $E_{12} = \frac{441}{691}E_4^3 + \frac{250}{691}E_6^2$.

Classical Gavan formula:

Then the fundamental classical formulas for the modular discriminant follow

$$\Delta(\tau) = \frac{1}{1728}\left(E_4^3(\tau) - E_6^2(\tau)\right) = \frac{1}{1728}\det\begin{pmatrix} \sqrt{7/3}E_4(\tau) & E_6(\tau) \\ E_6(\tau) & \sqrt{7/3}E_8(\tau) \end{pmatrix}.$$

The next formula is due to F. Garvan:

$$\Delta^2(\tau) = -\frac{691}{250\,(1728)^2}\det\begin{pmatrix} E_4(\tau) & E_6(\tau) & E_8(\tau) \\ E_6(\tau) & E_8(\tau) & E_{10}(\tau) \\ E_8(\tau) & E_{10}(\tau) & E_{12}(\tau) \end{pmatrix},$$

which was then proved and generalized in [Milne]. In this talk we give various generalizations for higher powers of the modular discriminant computed as a determinant of matrices containing deformed Weierstrass functions [Dong-Li-Mason, Mason-Tuite-Z].

The generalized Garvan formulas

Computations of the twisted partition function $Z_V^{(1)}\left[\begin{array}{c} f \\ g \end{array}\right](\tau)$ for the free fermion vertex operator superalgebra leads to two alternative expressions (see, e.g., [Kac, Mason-Tuite-Z]) as expansion over a basis:

$$Z_V^{(1)}\left[\begin{array}{c} f \\ g \end{array}\right](\tau) = q^{\kappa^2/2-1/24}\prod_{l\geqslant 1}(1-\theta^{-1}q^{l-\frac{1}{2}-\kappa})(1-\theta q^{l-\frac{1}{2}+\kappa}),$$

and

$$Z_V^{(1)}\left[\begin{array}{c} f \\ g \end{array}\right](\tau) = \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\vartheta^{(1)}\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right](0,\tau),$$

in terms of the torus theta series with characteristics:

$$\vartheta^{(1)}\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau) = \sum_{n\in\mathbb{Z}}\exp\left[i\pi(n+a)^2\tau + (n+a)(z+2\pi ib)\right].$$

Here we define $f = e^{2\pi i \alpha a(0)}$, $g = e^{2\pi i \beta a(0)}$, with some parameters α, $\beta \in \mathbb{R}$, and zero mode $a(0)$ of a Heisenberg subalgebra in the rank two free fermionic vertex operator superalgebra [Mason-Tuite-Z].

We also define $\phi = e^{-2\pi i \beta}$ and $\theta = e^{-2\pi i \alpha}$. Note that $Z_V^{(1)} \left[\begin{array}{c} f \\ g \end{array} \right] (\tau) = 0$ for $(\theta, \phi) = (1, 1)$, i.e., $(\alpha, \beta) \equiv (0, 0) \pmod{\mathbb{Z}}$.
Comparing two representations we obtain Jacobi triple product formula [Kac] which can be rewritten in the form:

$$\eta(\tau) = q^{-\kappa^2/2+1/24} e^{2\pi i (\alpha+1/2)(\beta+1/2)} \ \vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] (0, \tau) \ \det(1 - T_0),$$

where the determinant

$$\det(1 - T_0) = \left(\prod_{l \geqslant 1} (1 - \theta^{-1} q^{l - \frac{1}{2} - \kappa})(1 - \theta q^{l - \frac{1}{2} + \kappa}) \right)^{-1},$$

corresponds to sphere self-sewing to form a torus [Tuite-Z]. Thus we get the identity for the first power of the η-function.

Deformed Weierstrass functions

In [Dong-Li-Mason, Mason-Tuite-Z] the deformed Weierstrass functions (which can be expressed via deformed Eisenstein series) were defined and studied:

$$P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) = -\sum_{n\in\mathbb{Z}+\lambda}{}' \frac{q_z^n}{1-\theta^{-1}q^n} = \frac{1}{z} - \sum_{n\geqslant 1}\frac{1}{n}E_n\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau)z^{n-1},$$

for $q=e^{2\pi i\tau}$, and where $\sum'$ means we omit $n=0$ if $(\theta,\phi)=(1,1)$.

$$\begin{aligned} E_n\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau) \quad = \quad & -\frac{B_n(\lambda)}{n!} + \frac{1}{(n-1)!}\sum_{r\geqslant 0}{}' \frac{(r+\lambda)^{n-1}\theta^{-1}q^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}} \\ & + \frac{(-1)^n}{(n-1)!}\sum_{r\geqslant 1}\frac{(r-\lambda)^{n-1}\theta q^{r-\lambda}}{1-\theta q^{r-\lambda}}. \end{aligned}$$

Note that $P_2(z,\tau)=\wp(z,\tau)+E_2(\tau)$, $\phi=\exp(2\pi i\lambda)$ for $0\leqslant\lambda<1$. The Weierstrass $\wp$-function periodic in z with periods $2\pi i$ and $2\pi i\tau$ is $\wp(z,\tau)=z^{-2}+\sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)}\left[\frac{1}{(z-\omega_{m,n})^2}-\frac{1}{\omega_{m,n}^2}\right]$,

for $(z,\tau)\in\mathbb{C}\times\mathbb{H}$ with $\omega_{m,n}=2\pi i(m\tau+n)$.

In addition to that, for integers m_i, $n_j \geqslant 0$, satisfying $\sum_{i=1}^{r} m_i = \sum_{j=1}^{s} n_j$, let us introduce the notation

$$\Theta^{(1)}_{r,s,(m_i,n_i)}(x,y,\tau) = \frac{\prod\limits_{1\leqslant i\leqslant r, 1\leqslant j\leqslant s} \vartheta^{(1)}\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](x_i - y_j,\tau)^{m_i n_j}}{\prod\limits_{1\leqslant i<k\leqslant r} \vartheta^{(1)}\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](x_i - x_k,\tau)^{m_i m_k} \prod\limits_{1\leqslant j<l\leqslant s} \vartheta^{(1)}\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](y_j - y_l,\tau)^{n_j n_l}}.$$

Let us introduce $\mathbf{P}_n(\theta, \phi)$, a the $n \times n$ matrix:

$$\mathbf{P}_n(\theta, \phi) = \left[P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (x_i - y_j, \tau) \right], \quad (1 \leqslant i, j \leqslant n),$$

and another $(n+1) \times (n+1)$ matrix $\mathbf{Q}_n$:

$$\mathbf{Q}_n = \left(\begin{array}{cccc} P_1(x_1 - y_1, \tau) & \dots & P_1(x_1 - y_n, \tau) & 1 \\ \vdots & \ddots & & \vdots \\ P_1(x_n - y_1, \tau) & & P_1(x_n - y_n, \tau) & 1 \\ 1 & \dots & 1 & 0 \end{array} \right).$$

Proposition

Generalizing Garvan's formula, or $(\theta,\phi)\neq(1,1)$ *one has*

$$\Delta^n(\tau) = -\frac{\vartheta^{(1)}\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](0,\tau)\ \Theta^{(1)}_{8n,8n,(1,1)}(x,y,\tau).}{\vartheta^{(1)}\left[\begin{array}{c}-\beta+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{array}\right]\left(\sum_{i=1}^{8n}(x_i-y_i),\tau\right)}\ \det\left(\mathbf{P}_{8n}(\theta,\phi)\right),$$

for $(\theta,\phi)=(1,1)$,

$$\Delta^n(\tau) = i\ \frac{\Theta^{(1)}_{8n+1,8n+1,(1,1)}(x,y,\tau)}{\vartheta^{(1)}\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right]\left(\sum_{i=1}^{8n+1}(x_i-y_i),\tau\right)}\ \det\left(\mathbf{Q}_{8n+1}\right).$$

This can be also expressed in terms of deformed Eisenstein series by substitution of the definition of $P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau)$ in terms of $E_n\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau)$ leading to a quite involved formula which we do not give here.

Recall the genus one prime form $K^{(1)}(z,\tau)$ [Mumford, Fay]: $K^{(1)}(z,\tau) = -\frac{i}{\eta^3(\tau)}\,\vartheta^{(1)}\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](z,\tau)$. In [Mason-Tuite-Z] the elliptic function version of the Fay's generalized trisecant identity [Fay] was derived. For $(\theta,\phi)\neq(1,1)$ one has

$$\det\left(\mathbf{P}_n(\theta,\phi)\right) = \frac{\vartheta^{(1)}\left[\begin{array}{c}-\beta+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{array}\right]\left(\sum_{i=1}^{n}(x_i-y_i),\tau\right)}{\vartheta^{(1)}\left[\begin{array}{c}-\beta+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{array}\right](0,\tau)} \cdot\frac{\prod\limits_{1\leqslant i<j\leqslant n} K^{(1)}(x_i-x_j,\tau)K^{(1)}(y_i-y_j,\tau)}{\prod\limits_{1\leqslant i,j\leqslant n} K^{(1)}(x_i-y_j,\tau)},$$

and similarly for $(\theta,\phi)=(1,1)$,

$$\det\left(\mathbf{Q}_n\right) = -\frac{K^{(1)}\left(\sum_{i=1}^{n}(x_i-y_i),\tau\right)\prod\limits_{1\leqslant i<j\leqslant n} K^{(1)}(x_i-x_j,\tau)K^{(1)}(y_i-y_j,\tau)}{\prod\limits_{1\leqslant i,j\leqslant n} K^{(1)}(x_i-y_j,\tau)},$$

Higher power formulas

There exist also the analytic expansion [Mason-Tuite-Z], for $k, l \geqslant 1$,

$$P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z+z_1-z_2,\tau) = \sum_{k,l\geqslant 1} D\left[\begin{array}{c}\theta\\ \phi\end{array}\right](k,l,z)z_1^{k-1}z_2^{l-1},$$

$$D\left[\begin{array}{c}\theta\\ \phi\end{array}\right](k,l,z,\tau) = (-1)^{k+1}\binom{k+l-2}{k-1}P_{k+l-1}\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau,z).$$

Introduce is the block matrix

$$\mathbf{M}_{r,s} = \begin{pmatrix} \mathbf{D}^{(11)} & \dots & \mathbf{D}^{(1s)} \\ \vdots & \ddots & \vdots \\ \mathbf{D}^{(r1)} & \dots & \mathbf{D}^{(rs)} \end{pmatrix},$$

with $\mathbf{D}^{(ab)}$ the $m_a \times n_b$ matrix

$$\mathbf{D}^{(ab)}(i,j) = D\left[\begin{array}{c}\theta\\ \phi\end{array}\right](i,j,x_a-y_b,\tau), \quad (1\leqslant i\leqslant m_a, 1\leqslant j\leqslant n_b),$$

for $1 \leqslant a \leqslant r$ and $1 \leqslant b \leqslant s$.

Using the full version of the Fay's generalized trisecant identity [Mason-Tuite-Z], we derive the following

Proposition

For $(\theta, \phi) \neq (1,1)$, $\zeta = 8\Phi$,

$$\Delta^{\zeta}(\tau) = (-i)^{\Phi/24} \frac{\vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] (0, \tau) \, \Theta^{(1)}_{r,s,(m,n)}(x, y, \tau)}{\vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] \left(\sum_{i=1}^{r} m_i x_i - \sum_{j=1}^{s} n_j y_j, \tau \right)} \det(\mathbf{M}_{r,s}),$$

where $\Phi = \sum_{1 \leqslant i \leqslant r, 1 \leqslant k \leqslant s} m_i n_j - \sum_{1 \leqslant i < k \leqslant r} m_i m_k - \sum_{1 \leqslant j < l \leqslant s} n_j n_l$.

Genus two formulas

In [Tuite-Z] we derived the genus two counterpart of the triple Jacobi identity. In particular, for $\alpha = \beta = 1/2$ one has

$$Z^{(1)} \begin{bmatrix} f_{1/2} \\ g_{1/2} \end{bmatrix} (\tau) = K^{(1)}(z, \tau)/\eta^2(\tau),$$

where $\zeta(x) = \sum_{i=1}^{g} \partial_{z_i} \vartheta \left[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix} \right] (0, \Omega) \nu_i(x)$, $K^{(1)}(x - y, \tau) = \frac{\vartheta^{(1)} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] \left(\int_y^x \nu, \tau \right)}{\zeta(x)^{\frac{1}{2}} \zeta(y)^{\frac{1}{2}}}$.

In [Tuite-Z] we compared the rank two fermion partition function on a genus two Riemann surface

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau_1, \tau_2, \epsilon) = Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (\tau_1) \; Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (\tau_2) \; \det \left(I - Q^{(1)} \right)^{1/2},$$

with it's the bosonized version

$$Z_M^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau_1, \tau_2, \epsilon) = \frac{1}{\eta(\tau_1)\eta(\tau_2) \det \left(I - A_1 A_2 \right)^{1/2}} \vartheta^{(2)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\Omega^{(2)} \right),$$

with column vectors $\alpha = (\alpha_1, \alpha_2)^t$, $\beta = (\beta_1, \beta_2)^t$.

Here for $a = 1, 2$

$$Q^{(1)} = \begin{pmatrix} 0 & \xi F_1^{(1)} \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} \\ -\xi F_2^{(1)} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} & 0 \end{pmatrix},$$

$$F_a^{(1)} \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, l, \tau_a, \epsilon) = (-1)^l \epsilon^{\frac{1}{2}(k+l-1)} \binom{k+l-2}{k-1} E_{k+l-1}^{(1)} \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (\tau_a),$$

$$A_a(k, l, \tau_a, \epsilon) = \epsilon^{(k+l)/2} \frac{(-1)^{k+1}(k+l-1)!}{\sqrt{kl}(k-1)!(l-1)!} E_{k+l}^{(1)}(\tau_a),$$

which gives with $\alpha_{1/2} = (\alpha_1, 1/2)^t$, $\beta_{1/2} = (\beta_1, 1/2)^t$,

$$\begin{aligned} &\eta^3(\tau_1)\eta^3(\tau_2) \\ &= e^{2\pi i \alpha_{1/2}\cdot\beta_{1/2}} \frac{\left(K^{(1)}(z,\tau_1)K^{(1)}(z,\tau_2)\right)^2}{\vartheta^{(2)} \begin{bmatrix} \alpha_{1/2} \\ \beta_{1/2} \end{bmatrix} (\Omega^{(2)})} \det(I - A_1A_2)^{1/2} \det\left(I - Q^{(1)}\right) \end{aligned}$$

In particular, for $\tau = \tau_1 = \tau_2$ we obtain

$$\eta^6(\tau) = e^{2\pi i \alpha_{1/2}\cdot\beta_{1/2}} \frac{\left(K^{(1)}(z,\tau)\right)^4}{\vartheta^{(2)} \begin{bmatrix} \alpha_{1/2} \\ \beta_{1/2} \end{bmatrix} (\Omega^{(2)})} \det(I - A_1A_2)^{1/2} \det\left(I - Q^{(1)}\right).$$

The genus two: self-sewing formulas

In [Tuite-Z], by computing the genus two partition function for the fermionic vertex operator algebra and performing bosonization, we found a genus two analogue of the classical Jacobi triple product identity

$$\frac{\vartheta^{(2)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\Omega^{(2)}\right)}{\vartheta^{(1)}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\kappa w,\tau)} = e^{2i\pi\beta_2\kappa}\left(\frac{e^{i\pi B}\rho}{K^{(1)}(w,\tau)^2}\right)^{\frac{1}{2}\kappa^2} \det\left(I-T\right)\det\left(I-R\right)^{\frac{1}{2}}.$$

Here ρ is the torus self-sewing complex parameter, $\Omega^{(2)}$ is the genus two period matrix [Mason-Tuite], $-1/2<\kappa<1/2$, B is an odd integer parametrizing the formal branch cut, $T=\xi GD^{\theta_2}$,

$R_{ab}(k,l) = -\frac{\rho^{(k+l)/2}}{\sqrt{kl}}\begin{bmatrix}D(k,l,\tau,w) & C(k,l,\tau)\\ C(k,l,\tau) & D(l,k,\tau,w)\end{bmatrix}$,

$$C^{(1)}(k,l,\tau) = (-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}E^{(1)}_{k+l}(\tau),$$

$$D^{(1)}(k,l,z,\tau) = (-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}P^{(1)}_{k+l}(\tau,z),$$

$D^{\theta_2}(k,l) = \left[\begin{array}{cc} \theta_2^{-1} & 0 \\ 0 & -\theta_2 \end{array} \right] \delta(k,l)$, $\xi \in \{\pm\sqrt{-1}\}$, and an infinite diagonal matrix:

$$G = \left[\frac{\rho^{\frac{1}{2}(k_a+l_b-1)}}{(2\pi i)^2} \oint_{\mathcal{C}_{\overline{a}}(x_{\overline{a}})} \oint_{\mathcal{C}_b(y_b)} (x_{\overline{a}})^{-k_a} (y_b)^{-l_b} S_\kappa(x_{\overline{a}}, y_b) \; dx_{\overline{a}}^{\frac{1}{2}} \; dy_b^{\frac{1}{2}} \right].$$

The genus two Szegő kernel for x, y considered on the torus is given by

$$\begin{aligned} S^{(2)}(x,y) &= S_\kappa^{(2)}(x,y) + \xi h(x) D^\theta (I-T)^{-1} \overline{h}^t(y) \\ &= \left[(x-y)^{-1} + \sum_{k,l} E_{k,l}^{(2)}(\Omega^{(2)}) x^{-k} y^l \right] dx^{\frac{1}{2}} dy^{\frac{1}{2}}, \end{aligned}$$

where $\overline{h}^t(y)$ denotes the transpose to

$\overline{h}(y) = (\rho^{\frac{1}{2}(k_a - \frac{1}{2})} \frac{1}{2\pi i} \oint_{\mathcal{C}_a(y_a)} y_a^{-k_a} S_\kappa^{(2)}(x, y_a) dy_a^{\frac{1}{2}})$, $\vartheta_1(z,\tau) = \vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (z,\tau)$.

$$S_\kappa^{(2)}(x,y) = \left(\frac{\vartheta_1(x-w,\tau)\vartheta_1(y,\tau)}{\vartheta_1(x,\tau)\vartheta_1(y-w,\tau)} \right)^\kappa \frac{\vartheta \left[\begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right] (x-y+\kappa w, \tau)}{\vartheta \left[\begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right] (\kappa w, \tau) K(x-y,\tau)} dx^{\frac{1}{2}} dy^{\frac{1}{2}}.$$

Let us introduce also the semi-infinite matrices $H = ((h(x_i))(k,a))$, $\overline{H}^t = \left(\left(\overline{h}(y_i)\right)(l,b)\right)^t$, with n row indexed by i and columns indexed by $k \geqslant 1$ and $a = 1, 2$ and $\overline{H}^t$ is semi-infinite with rows indexed by $l \geqslant 1$ and $b = 1, 2$ and with n columns indexed by j.
Then we obtain [Levin-Shin-Z]

Proposition

For $n \geqslant 1$, $w \in \mathbb{C}$, $a = 1, 2$ a genus two generalization of the Garvan's formula is

$$\eta^{3\kappa^2}(\tau) = \frac{e^{-2i\pi\beta_2\kappa}\vartheta^{(2)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\Omega^{(2)}\right)\vartheta^{(1)}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,\tau)^{\kappa^2}\det\left[S^{(2)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\mathbf{x},\mathbf{y})\right]}{\left(-e^{i\pi B}\rho\right)^{\frac{1}{2}\kappa^2}\vartheta^{(1)}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\kappa w,\tau)\det(I-R)^{\frac{1}{2}}\det\begin{bmatrix}S_\kappa^{(2)} & -\xi H D^{\theta_2}\\ \overline{H}^t & I-T\end{bmatrix}}$$

□

We are looking forward for higher genus Eisenstein series.

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