Cohomology for Linearized Ricci Curvature

Prague Cohomology Seminar

Roee Leder, 29.04.2026, HUJI

- L., **Hodge Theory for linearized boundary-value problems on general geometric structures,** arXiv:2504.18494
- L., **Cohomology for linearized Ricci curvature,** arXiv:2510.12797

Ricci curvature equations

Given a source term $T \in S^2_M$, the ***Ricci curvature equations***, for an unknown $g \in \mathcal{M}_M$, are the covariant system of equations:

$$(*) \quad \mathrm{Ric}_g = T$$

- (M, g) is a compact Riemannian with boundary (which might be empty).
- $\dim M \geq 3$.
- S^2_M the space of smooth symmetric tensor fields over M.
- $\mathcal{M}_M \subset S^2_M$ the open subset of Riemannian metrics over M.
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Goal: Solvability and uniqueness conditions for $(*)$. ***Very Hard***

Symmetries and constraints

For every $g \in \mathcal{M}_M$ and a diffeomorphism $\phi \in \mathcal{D}_M$:

$$\mathrm{Ric}_{\phi^* g} = \phi^* \mathrm{Ric}_g, \qquad \delta_g \mathrm{B}_g \mathrm{Ric}_g = 0$$

Here, $\delta_g \mathrm{B}_g : S^2_M \to \mathfrak{X}_M$ is the **Bianchi operator**:

- $\mathrm{B}_g : S^2_M \to S^2_M$ is the isomorphism $\mathrm{B}_g = \mathrm{Id} - \frac{1}{2}\mathrm{tr}_g(\cdot) g$;
- $\delta_g : S^2 \to \mathfrak{X}_M$ is tensor divergence, $\delta_g \sigma \coloneqq -\sum_i \nabla_{E_i} \sigma(E_i, \cdot)$;

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Nontrivial (and unclear) solvability and uniqueness conditions for $\mathrm{Ric}_g = T$:

- Not every T satisfies $\delta_g \mathrm{B}_g T = 0$ (a-priori depends on the solution g);
- If g is a solution, then $\phi^* g$ is also a solution whenever $\phi^* T = T$ (if such exists).

Linearized Ricci curvature equations

Linearization: Given a source term $T \in S^2_M$, for an unknown $\sigma \in S^2_M$,

$$\mathrm{DRic}_g \sigma = T, \qquad \text{where} \qquad \mathrm{DRic}_g \sigma \coloneqq \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ric}_{g+t\sigma}$$

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Despite extensive research, ***resisted*** analysis without restrictive assumptions:

- Global topological assumptions (e.g, $M \simeq \mathbb{T}^n, \mathbb{R}^n, \mathbb{S}^n$).
- Vanishing curvature assumptions (e.g., $\mathrm{Rm}_g = 0, \mathrm{Ric}_g = 0, \mathrm{Ric}_g = \lambda g \ldots$).
- Analyticity of M, g, T, or non-singularity of T.

DeTurck '81, Hamilton '84, Goldschmidt '99, Anderson '08, An et al '22, Hintz '24,

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- **Goal:** Remove any structural assumptions on (M, g) by reducing the entire problem to the study of a newly identified **cohomology**.

The cohomological picture

Observation: under assumptions on g, the linearization of the identities:

$$\mathrm{Ric}_{\phi^* g} = \phi^* \mathrm{Ric}_g, \qquad \delta_g \mathrm{B}_g \mathrm{Ric}_g = 0$$

Does lead to uniqueness and solvability conditions for the linearized problem:

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Specifically, if $\mathrm{Ric}_g = 0$:

$$\mathrm{DRic}_g \delta_g^* X = 0 \quad \text{on} \quad X \in \mathfrak{X}_M, \qquad \delta_g \mathrm{B}_g \mathrm{DRic}_g \sigma = 0 \quad \text{on} \quad \sigma \in S_M^2$$

- $\delta_g^* X = \frac{1}{2} \mathcal{L}_X g$ is the Killing operator.

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- $\delta_g^* X = \frac{1}{2} \mathcal{L}_X g$ is the Killing operator.

$$\mathcal{E}_M^1(g) \simeq \ker \mathrm{DRic}_g / \mathrm{Im}\, \delta_g^*, \qquad \mathcal{E}_M^2(g) \simeq \ker \delta_g \mathrm{B}_g / \mathrm{Im}\, \mathrm{DRic}_g$$

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Observation: under $\mathrm{Ric}_g = 0$:

$$\mathcal{E}^1_M(g) \simeq \ker \mathrm{DRic}_g / \mathrm{Im}\, \delta^*_g\,, \qquad \mathcal{E}^2_M(g) \simeq \ker \delta_g \mathrm{B}_g / \mathrm{Im}\, \mathrm{DRic}_g$$

Does lead to uniqueness and solvability conditions for the linearized problem:

$$(*) \quad \mathrm{DRic}_g \sigma = T$$

*The problem $(*)$ is **solvable** if and only if $\delta_g \mathrm{B}_g T = 0$ and $T \perp \mathcal{E}^2_M(g)$.*

*The solution is **unique** modulo $\mathcal{E}^1_M(g)$ and the gauge freedom $X \mapsto \sigma + \delta^*_g X$.*

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- Hintz ('24): when (M, g) is hyperbolic spacetimes, $\mathcal{E}^2_M(g)$ isomorphic to $\ker \delta_g^*$
- Anderson school ('08): when $\pi_1(M, \partial M) = \{0\}$, with the natural boundary conditions, $\mathcal{E}^1_M(g) = \{0\}$.

The cohomological picture

Observation: under $\mathrm{Ric}_g \neq 0$, the linearization of the identities:

$$\mathrm{DRic}_g \delta_g^* X \neq 0 \quad \text{on} \quad X \in \mathfrak{X}_M, \qquad \delta_g \mathrm{B}_g \mathrm{DRic}_g \sigma \neq 0 \quad \text{on} \quad \sigma \in S_M^2$$

Does not lead to uniqueness and solvability conditions for the linearized problem:

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~~*The problem $(*)$ is **solvable** if and only if $\delta_g \mathrm{B}_g T = 0$ and $T \perp \mathcal{E}_M^2(g)$.*~~

~~*The solution is **unique** modulo $\mathcal{E}_M^1(g)$ and the gauge freedom $X \mapsto \sigma + \delta_g^* X$.*~~

?

Outline for the talk

- **Goal:** Remove any structural assumptions on (M, g), at least in the interior, by reducing the entire problem to the study of a newly identified **cohomology**.
- Generalized Hodge theory we have developed to overcome such restrictions, based on the **Boutet de Monvel calculus**.
- Cast the problem inside this Hodge theory.
- Present analysis of geometric aspects of the newly identified cohomology, using **Bochner technique**.

Accurate Setup

$$0 \longrightarrow \mathfrak{X}_M \xrightarrow{\delta_g^*} S_M^2 \xrightarrow{\mathrm{DRic}_g} S_M^2 \xrightarrow{\delta_g \mathrm{B}_g} \mathfrak{X}_M \longrightarrow 0$$

Accurate Setup

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Supplement the problem with **divergence-free gauge** and **Cauchy data**:

$$\mathrm{DRic}_g \sigma = T, \qquad \delta_g \sigma = 0 \qquad \text{in } \ M$$

$$\mathbb{P}^{\mathrm{tt}} \sigma = 0, \qquad \mathrm{DA}_g \sigma = 0 \qquad \text{on } \ \partial M$$

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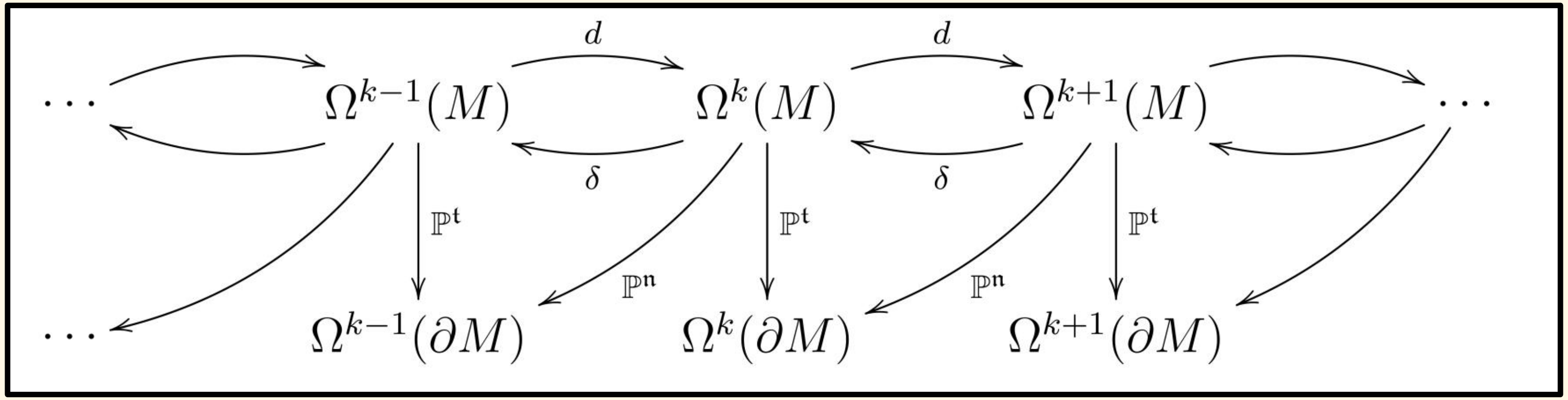
We shall now examine Hodge theory and find exactly where it fails in producing solvability and uniqueness conditions for the above.

Hodge theory

Atiyah & Singer ('70). Many introductions and generalizations: Schulze '82, Schulze & Seiler '19, Tarkhanov et al '07, Wallenta '15.

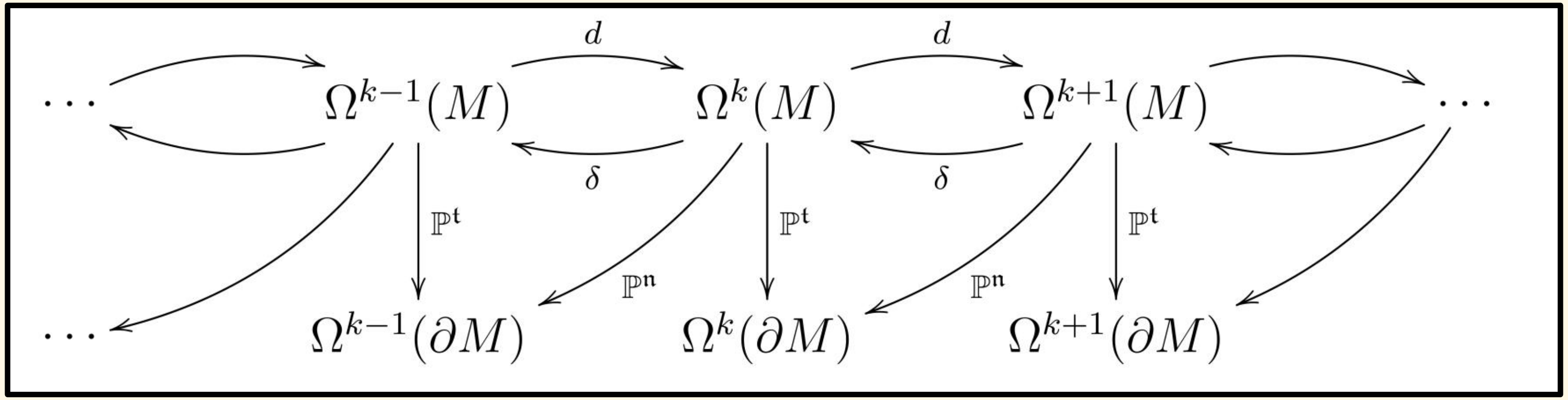
We follow a slightly different introduction (Taylor, '11).

The de-Rham Complex



$$dd = 0 \qquad \int_M (d\eta, \rho)_g dV = \int_M (\eta, \delta\rho)_g dV + \int_{\partial M} \left(\mathbb{P}^{\mathfrak{t}}\eta, \mathbb{P}^{\mathfrak{n}}\rho\right)_{g_\partial} dS$$

The de-Rham Complex



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$$\langle d\eta, \rho \rangle_{L^2(M)} = \langle \eta, \delta\rho \rangle_{L^2(M)} + \left\langle \mathbb{P}^{\mathfrak{t}}\eta, \mathbb{P}^{\mathfrak{n}}\rho \right\rangle_{L^2(\partial M)}$$

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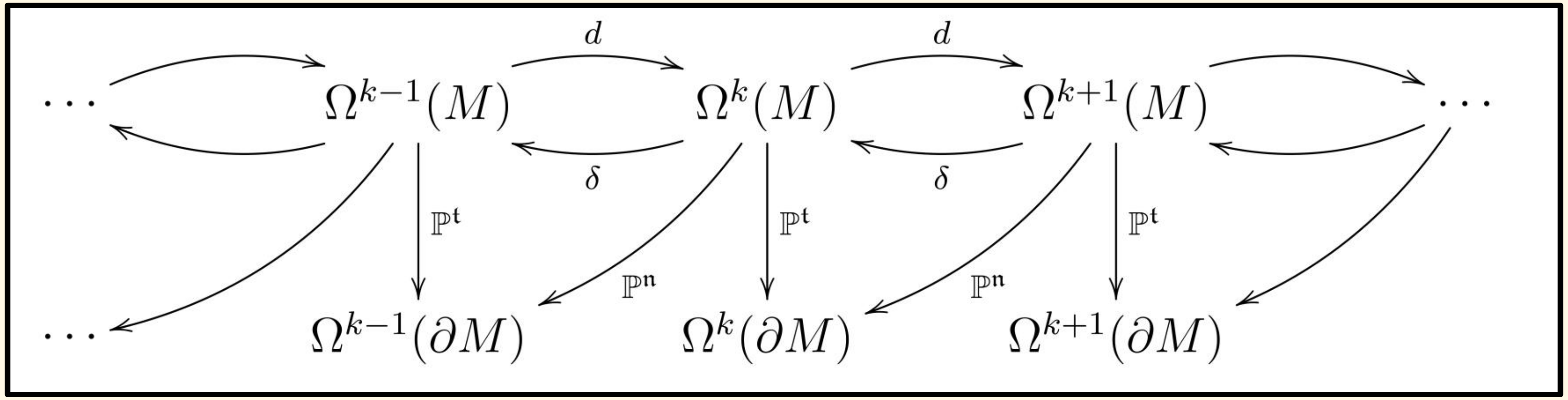

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Theorem (Elliptic Problem): The Hodge Laplacian,

$$\Delta = d\delta + \delta d$$

is **elliptic** when supplemented with either:

$$\mathbb{P}^{\mathrm{t}} \oplus \mathbb{P}^{\mathrm{t}}\delta \quad \text{(Dirichlet)}, \qquad \mathbb{P}^{\mathrm{n}} \oplus \mathbb{P}^{\mathrm{n}} d \quad \text{(Neumann)}$$

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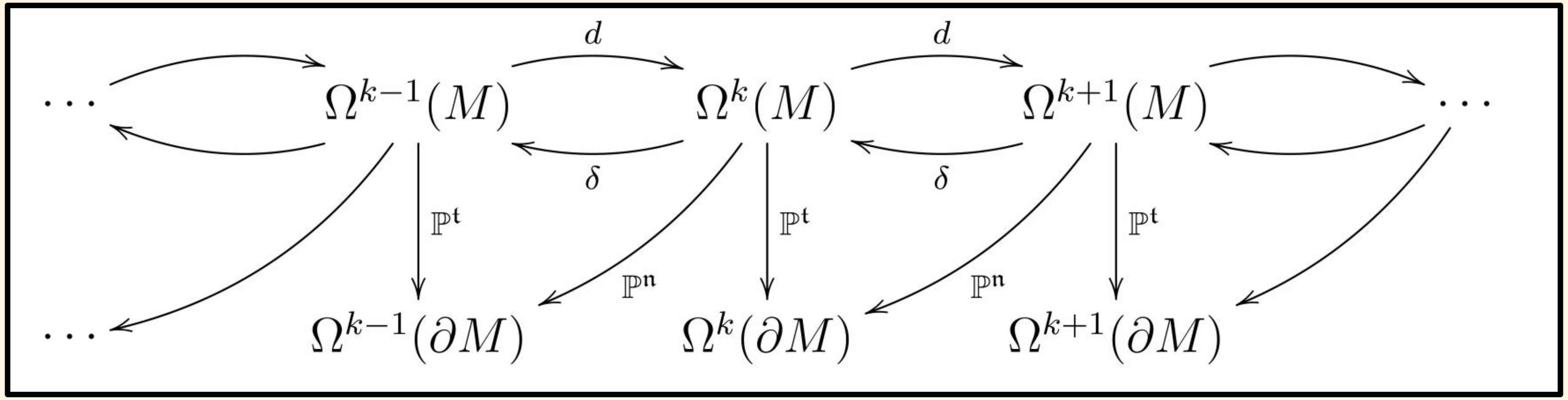

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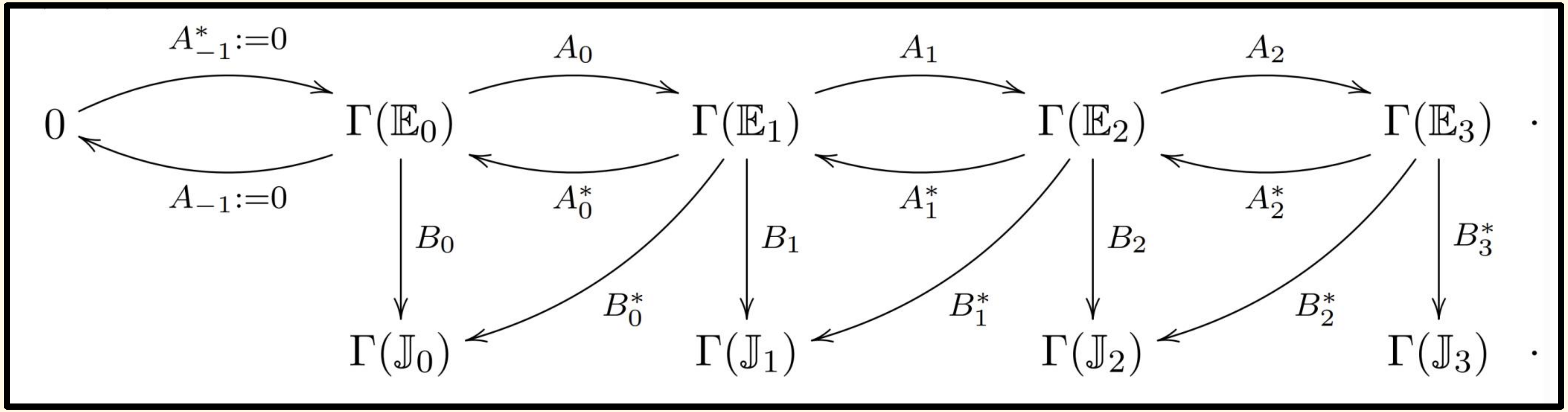
Theorem (Hodge decomposition, Dirichlet): there is an L^2-orthogonal, topologically direct decomposition of Fréchet spaces:

$$\Omega_M^k = \mathrm{Im}\left(d|_{\ker \mathbb{P}^{\mathfrak{t}}}\right) \oplus \mathcal{H}_{\mathrm{D}}^{\alpha+1} \oplus \mathrm{Im}\,\delta$$

Where $\mathcal{H}_{\mathrm{D}}^{\alpha+1} \coloneqq \ker\left(d, \delta, \mathbb{P}^{\mathfrak{t}}\right) \simeq \ker\left(d|_{\ker \mathbb{P}^{\mathfrak{t}}}\right)/\mathrm{Im}\left(d|_{\ker \mathbb{P}^{\mathfrak{t}}}\right)$.

Generalization: Elliptic complexes



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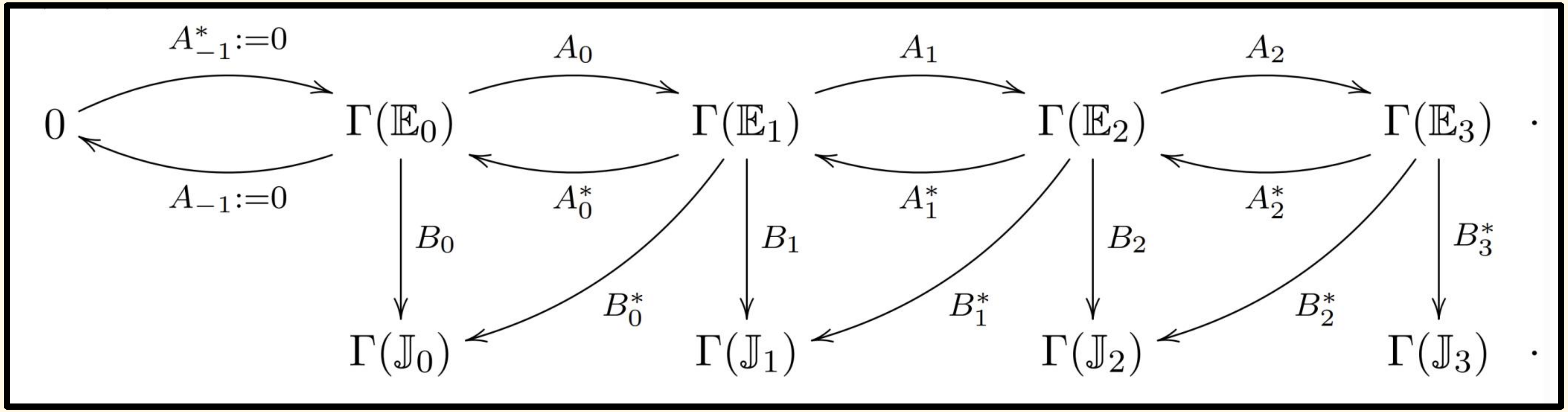

- **Cochain complex**: $A_{\alpha+1}A_\alpha = 0.$
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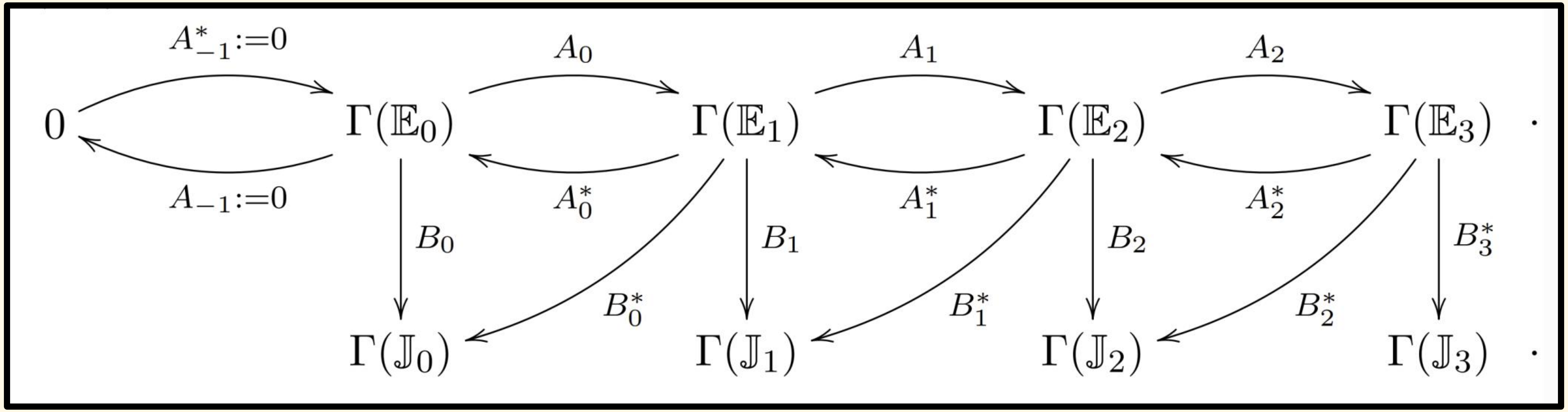

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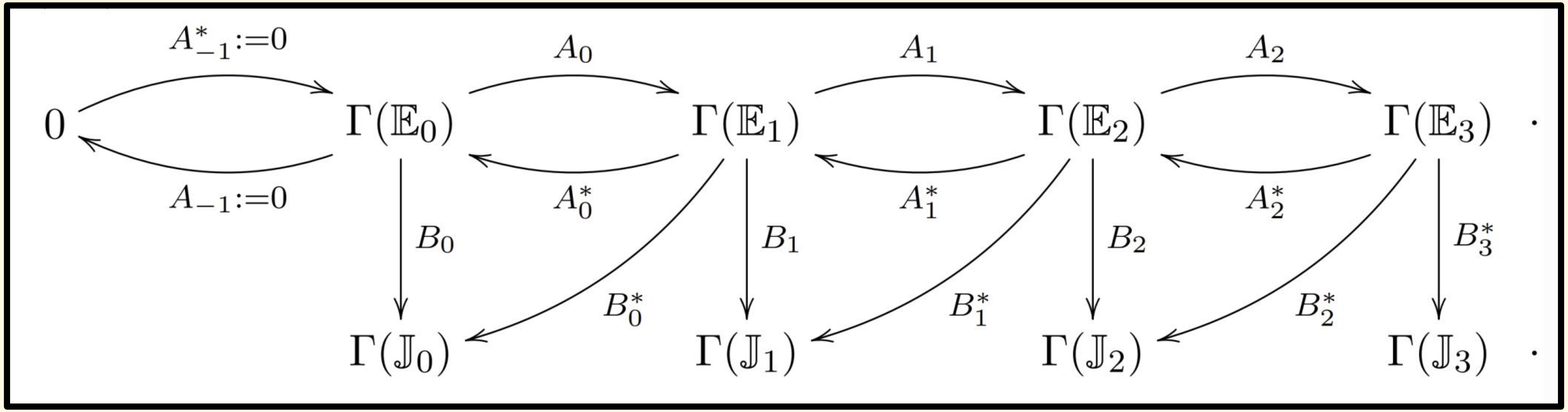
Where $\mathcal{H}_{\mathrm{D}}^{\alpha+1} = \ker(A_{\alpha+1}, A^*_\alpha, B_\alpha) \simeq \ker\left(A_{\alpha+1}|_{\ker B_{\alpha+1}}\right)/\mathrm{Im}\left(A_\alpha|_{\ker B_\alpha}\right)$.

Problem: not many valid examples



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Example: Linearized Ricci curvature

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Order Reduction

(DeTruck 82', Hamilton 84') Linearizing, for every $g \in \mathcal{M}_M$ and $\phi \in \mathcal{D}_M$:

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Does lead to the following:

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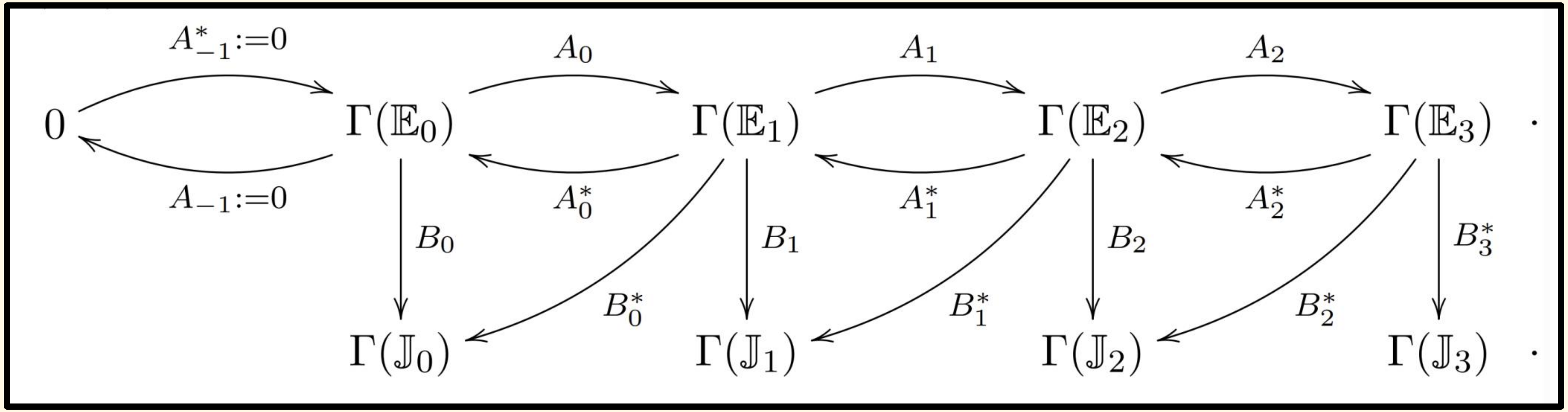
Observation: this order reduction allows us to cast the sequence:

$$0 \longrightarrow \mathfrak{X}_M \xrightarrow{\delta_g^*} S_M^2 \xrightarrow{\mathrm{DRic}_g} S_M^2 \xrightarrow{\delta_g \mathrm{B}_g} \mathfrak{X}_M \longrightarrow 0$$

Into an *almost* cochain complex, and *lift* the operators in a meaningful way, by using a form of *generalized Hodge theory.*

Elliptic pre-complexes (Dirichlet conditions)



- **Order reduction**: $\operatorname{ord}(A_{\alpha+1}A_\alpha) \le \operatorname{ord}(A_\alpha)$ and $\ker B_{\alpha-1} \subseteq \ker B_\alpha A_{\alpha-1}$
- **Green's formula**: for all $\psi \in \Gamma(\mathbb{E}_\alpha)$, $\eta \in \Gamma(\mathbb{E}_{\alpha+1})$, with normal operators

$$\langle A_\alpha \psi, \eta \rangle_{L^2(M)} = \langle \psi, A_\alpha^* \eta \rangle_{L^2(M)} + \langle B_\alpha \psi, B_\alpha^* \eta \rangle_{L^2(\partial M)}.$$

- **Overdetermined ellipticity in the Douglas-Nirenberg sense**:

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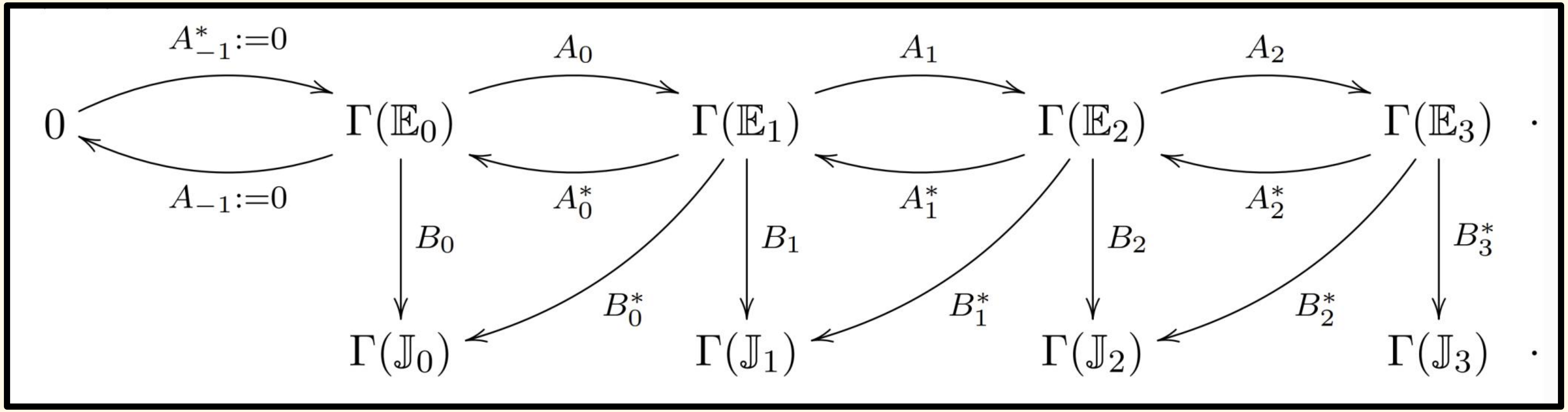

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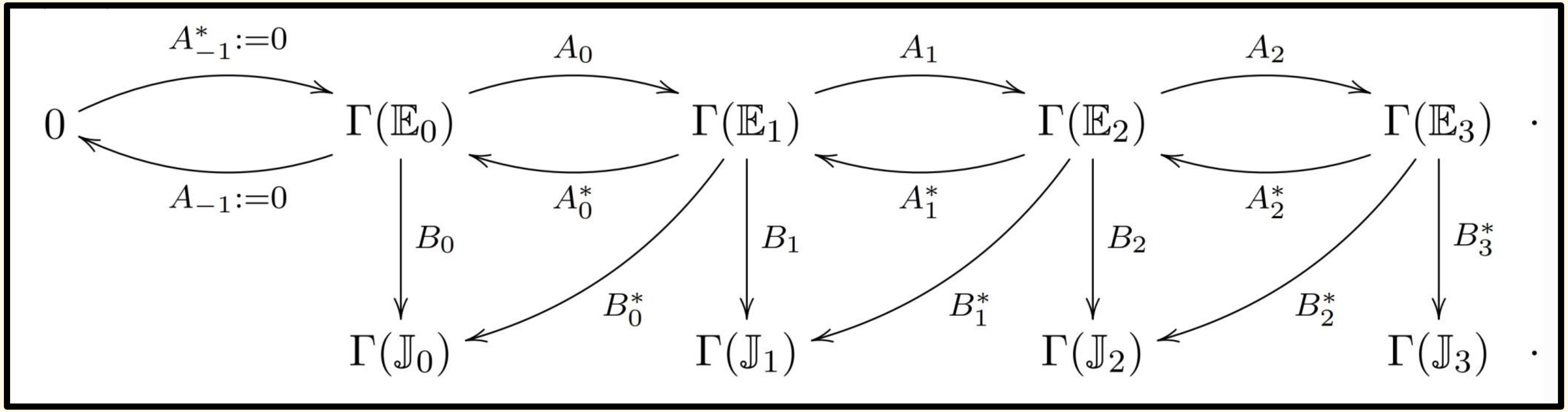

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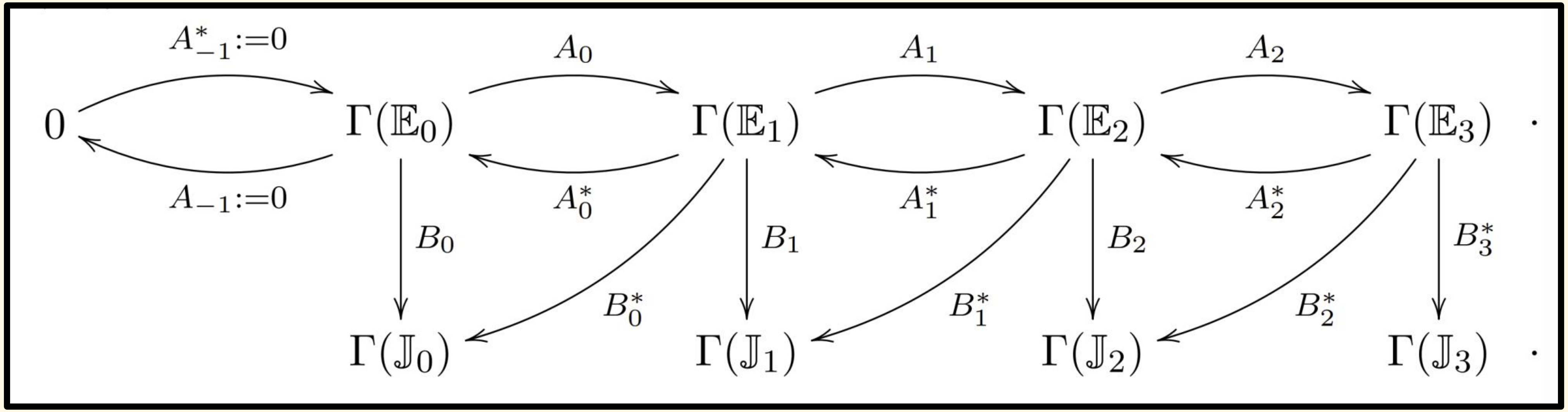
- **Green's formula**: for all $\psi \in \Gamma(\mathbb{E}_\alpha)$, $\eta \in \Gamma(\mathbb{E}_{\alpha+1})$, **with normal operators**

$$\langle A_\alpha \psi, \eta \rangle_{L^2(M)} = \langle \psi, A_\alpha^* \eta \rangle_{L^2(M)} + \langle B_\alpha \psi, B_\alpha^* \eta \rangle_{L^2(\partial M)}.$$

- **Overdetermined ellipticity in the Douglas-Nirenberg sense**:

$$A_\alpha \oplus A^*_{\alpha-1} \oplus B_\alpha \quad \text{(Dirichlet)}$$

Elliptic pre-complexes (Dirichlet conditions)



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Example: Linearized Ricci curvature

$$0 \longrightarrow \mathfrak{X}_M \xrightarrow{\delta_g^*} S_M^2 \xrightarrow{\mathrm{DRic}_g} S_M^2 \xrightarrow{\delta_g \mathrm{B}_g} \mathfrak{X}_M \longrightarrow 0$$

Supplement the problem with **divergence-free gauge** and **Cauchy data**:

$$\mathrm{DRic}_g \sigma = T, \qquad \delta_g \sigma = 0 \qquad \text{in} \quad M$$

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The lifted complex

Theorem (Lifted complex, L. '25): Let $(A_\bullet)$ be an elliptic pre-complex (based on Dirichlet conditions). Then there exist continuous linear maps of Fréchet spaces,

$$\mathcal{A}_{\alpha+1}\colon \Gamma(\mathbb{E}_{\alpha+1}) \to \Gamma(\mathbb{E}_{\alpha+2})$$

uniquely characterized by the following properties:

1. $\mathcal{A}_{\alpha+1}\mathcal{A}_\alpha = 0, \quad$ on $\quad \ker B_\alpha$,
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Proposition (L. '25): As a byproduct of this functional analytic characterization,

$$\mathcal{A}_\alpha - A_\alpha$$

is in the Boutet-de-Monvel calculus, of order and class zero, hence admits adjoints.

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Theorem (Hodge decomposition, L. '25): there is a topologically-direct decomposition

$$\Gamma(\mathbb{E}_{\alpha+1}) = \operatorname{Im}\left(\mathcal{A}_\alpha|_{\ker B_\alpha}\right) \oplus \mathcal{H}_{\mathrm{D}}^{\alpha+1} \oplus \operatorname{Im} \mathcal{A}_{\alpha+1}^*,$$

where $\mathcal{H}_{\mathrm{D}}^{\alpha+1} = \ker(A_{\alpha+1}, P_\alpha A_\alpha^*, B_\alpha) \simeq \ker\left(\mathcal{A}_{\alpha+1}|_{\ker B_{\alpha+1}}\right)/\operatorname{Im}\left(\mathcal{A}_\alpha|_{\ker B_\alpha}\right)$

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Hodge theory for Ricci curvature

$$0 \longrightarrow \mathfrak{X}_M \xrightarrow{\delta_g^*} S_M^2 \xrightarrow{\mathrm{DRic}_g} S_M^2 \xrightarrow{\delta_g \mathrm{B}_g} \mathfrak{X}_M \longrightarrow 0$$

Supplement the problem with divergence-free gauge and Cauchy data:

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Theorem (L. '25, Uniqueness): There exists lifted systems,

$$\mathcal{DRic}_g : S_M^2 \to S_M^2, \qquad \boldsymbol{\delta}_g : S_M^2 \to \mathfrak{X}_M$$

Such that, by defining $\mathcal{E}_M^1(g) \coloneqq \ker\big(\mathrm{DRic}_g, \delta_g, \mathbb{P}^{\mathrm{tt}}, \mathrm{DA}_g\big)$, we have:

$$\mathcal{E}_M^1(g) \simeq \ker\left(\mathcal{DRic}_g|_{\ker(\mathbb{P}^{\mathrm{tt}}, \mathrm{DA}_g)}\right) / \mathrm{Im}\big(\delta_g^*|_{\ker(\cdot)|_{\partial M}}\big).$$

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Theorem (L. '25, Solvability): (For simplicity, $\dim M = 3$) There exists lifted systems,

$$\mathcal{DRic}_g : S^2_M \to S^2_M, \qquad \boldsymbol{\delta}_g : S^2_M \to \mathfrak{X}_M$$

Such that, by defining $\mathcal{E}^2_M(g) \coloneqq \ker\left(\mathrm{DRic}_g, \delta_g, \mathbb{P}^{\mathfrak{n}}\right)$, we have:

$$\mathcal{E}^2_M(g) \simeq \ker\left(\boldsymbol{\delta}_g \mathrm{B}_g|_{\ker(\mathbb{P}^{\mathfrak{n}})}\right) / \mathrm{Im}\left(\mathcal{DRic}_g|_{\ker(\mathbb{P}^{\mathrm{tt}}, \mathrm{DA}_g)}\right).$$

The problem admits a solution if and only if:

$$\boldsymbol{\delta}_g \mathrm{B}_g T = 0, \qquad \mathbb{P}^{\mathfrak{n}} \mathrm{B}_g T = 0, \qquad \mathrm{B}_g T \perp_{L^2} \mathcal{E}^2_M(g)$$

Outline for the talk

- ✓ **Goal:** Remove any structural assumptions on (M, g), at least in the interior, by reducing the entire problem to the study of a newly identified **cohomology**.
- ✓ Generalized Hodge theory we have developed to overcome such restrictions, based on the **Boutet de Monvel calculus**.
- ✓ Cast the problem inside this Hodge theory.
- Present analysis of geometric aspects of the newly identified cohomology, using **Bochner technique**.

Bochner Technique (1)

For every $\sigma \in \mathcal{E}^1(g) = \ker\left(\mathrm{DRic}_g, \delta_g, \mathbb{P}^{\mathrm{tt}}, \mathrm{DA}_g\right)$:

$$\mathrm{DRic}_g \sigma - \frac{1}{2}\nabla^*\nabla\sigma = -\frac{1}{2} H_g \mathrm{tr}_g \sigma + \frac{1}{2}\mathcal{R}_g$$

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Bocher Formula (1) (L. '25): For every $\sigma \in \mathcal{E}^1_M(g)$, set:

$$v := \mathbb{P}^{\mathrm{nt}}_g\sigma, \qquad \phi := \mathbb{P}^{\mathrm{nn}}_g\sigma, \qquad \tau := \mathrm{tr}_g\sigma$$

Let $\mathcal{R}_g$ be the Lichnerowicz curvature operator, H_g the mean curvature of the boundary. Then the following formula holds:

$$\int_M \left[|\nabla\sigma|^2 + |d\tau|^2 + \big(\mathcal{R}_g\sigma,\sigma\big)\right] d\mathrm{Vol}_g = \int_{\partial M}\left[\mathrm{H}_g\phi^2 + 2\mathrm{H}_g|v|^2 + 2\mathrm{A}_g(v;v)\right] d\mathrm{Vol}_{g_\partial}$$

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Corollary: if $\mathcal{R}_g \geq 0$ and $\mathrm{A}_g < 0$ then$\mathcal{E}^1(g) = 0$.

Hodge theory for Perturbed Ricci curvature

$$0 \longrightarrow \mathfrak{X}_M \xrightarrow{\ \delta_g^*\ } S_M^2 \xrightarrow{\ \mathrm{D}_\Gamma \mathrm{Ric}_g := \mathrm{DRic}_g + \Gamma\ } S_M^2 \xrightarrow{\ \delta_g \mathrm{B}_g\ } \mathfrak{X}_M \longrightarrow 0$$

Supplement the problem with divergence-free gauge and Cauchy data:

$$\mathrm{D}_\Gamma \mathrm{Ric}_g \sigma = T, \qquad \delta_g \sigma = 0 \qquad \text{in } M$$

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Theorem (L. '25, Uniqueness): There exists lifted systems,

$$\mathcal{D}_\Gamma \mathcal{R}ic_g : S_M^2 \to S_M^2, \qquad \boldsymbol{\delta}_g : S_M^2 \to \mathfrak{X}_M$$

Such that, by defining $\mathcal{E}_M^1(g,\Gamma) = \ker\big(\mathrm{D}_\Gamma \mathrm{Ric}_g, \delta_g, \mathbb{P}^{\mathrm{tt}}, \mathrm{DA}_g\big)$, we have:

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Corollary: if $\mathcal{R}_g + \Gamma \geq 0$ and $A_g < 0$ then $\mathcal{E}^1(g,\Gamma) = 0$.

Bochner Technique – Perturbed (2)

$(\dim M = 3)$. For every $\sigma \in \ker\left(\mathrm{D}_\Gamma \mathrm{Ric}_g, \delta_g, \mathbb{P}^{\mathfrak{n}}\right)$:

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Bocher Formula (2) (L. '25): Suppose $(\partial M, g_\partial)$ is round and $\mathrm{A}_g = -\sqrt{\kappa}\, g_\partial$.

Let $\eta \in \mathcal{E}_M^2(g, \Gamma)$ and set:

$$\xi \coloneqq \mathbb{P}^{\mathrm{tt}}\nabla_{\partial_r}\eta, \qquad \mu \coloneqq \mathbb{P}^{\mathrm{tt}}\eta$$

Then there exists a $\chi \in C^\infty_{\partial M}$ such that $\xi = H_{g_\partial}\chi + \kappa\chi$, and following formula holds:

$$\int_M \left[|\nabla\eta|^2 + \left(\left(\mathcal{R}_g + \Gamma\right)\eta, \eta\right)\right] d\mathrm{Vol}_g = \sqrt{\kappa}\int_{\partial M} -|d\chi|^2 + 2\kappa|\chi|^2 - \left(\mathrm{tr}_{g_\partial}\xi\right)\left(\mathrm{tr}_{g_\partial}\mu\right) d\mathrm{Vol}_{g_\partial}$$

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Corollary: If $\mathcal{R}_g + \Gamma \geq 0$ and $\mathrm{tr}_g \sigma = 0$ then $\sigma = 0$.