Wigner - Weyl calculus and the theory of topological response

A review talk

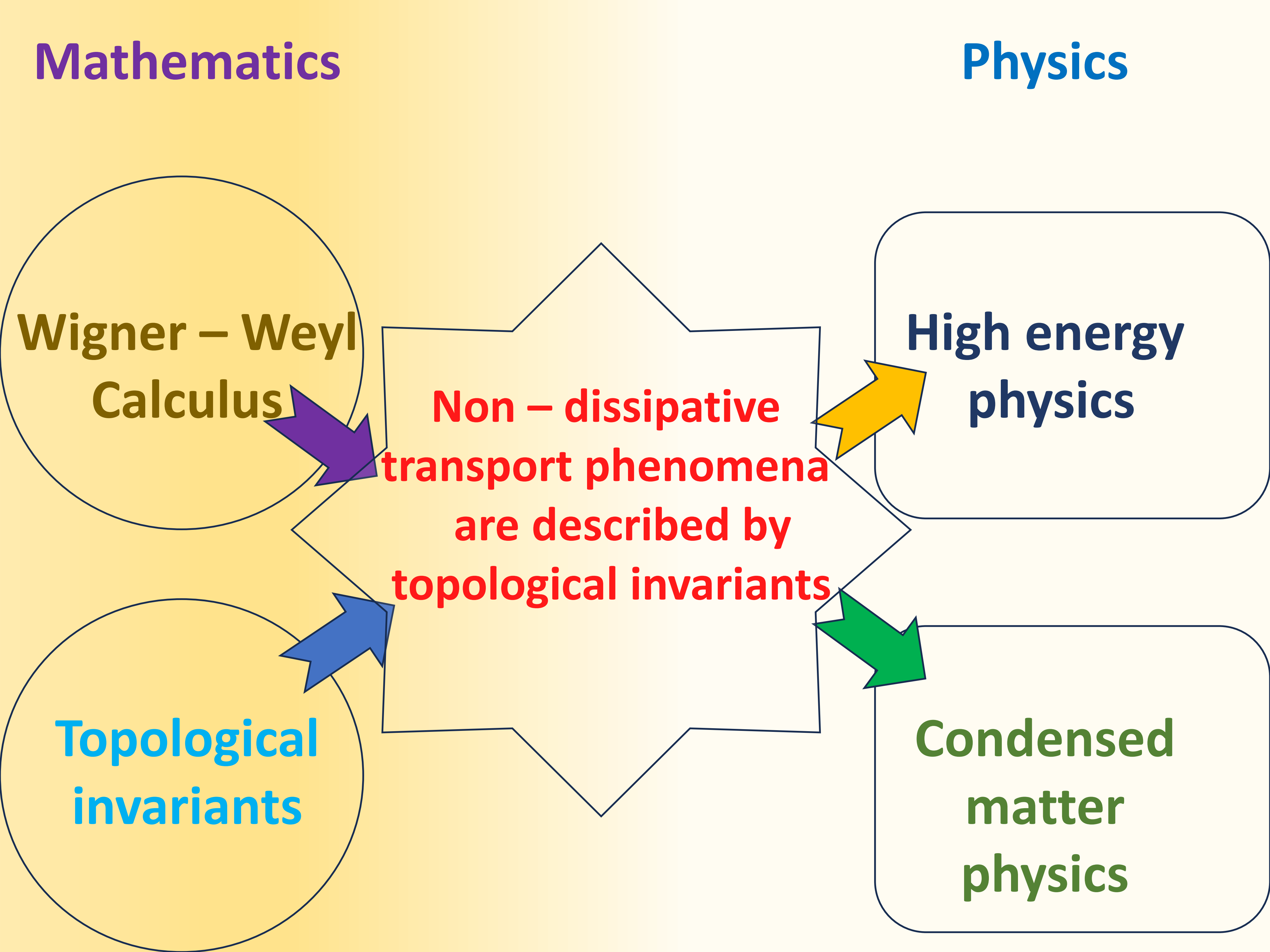
M.A. Zubkov
Ariel University Israel

Seminar

19 November 2025

Institute of Mathematics of ASCR, Žitná 25, Praha 1,

Cohomology in algebra, geometry, physics and statistics



Mathematics
Physics
Wigner – Weyl Calculus
Topological invariants
Non – dissipative transport phenomena are described by topological invariants
High energy physics
Condensed matter physics

collaborators:

R.Abramchuk (Ariel University, Israel)
C.Banerjee (Hong – Kong University, China)
I.Fialkovsky (ABC, Sao Paolo, Brazil)
M.Lewkowicz (Ariel University, Israel)
M.Selch (Ariel University)
M.Suleymanov (Bar Ilan University, Israel)
C.Zhang (Wuerzburg University, Germany)
X.Wu (Henan Normal University, China)
R.Chobanyan (Ariel University, Israel)
Z.Khaidukov (MIPT, Russia)
P.D.Xavier (Ariel University, Israel)

- *Zubkov, M. A., and Xi Wu. Annals of Physics 418 (2020): 168179.*
- *C.X. Zhang, M.A. Zubkov*
- *Journal of Physics A: 53 (19), 195002 (2020)*
- *C.X. Zhang, M.A. Zubkov Annals of Physics 444, 169016 (2022)*
- *M.Suleymanov, M.Zubkov, Physical Review D 102 (7), 076019 (2020)*
- *M.Zubkov, R.Abramchuk Physical Review D 107 (9), 094021 (2023)*
- *M A Zubkov 2024 J. Phys.: Condens. Matter 36 415501*
- *Abramchuk, Ruslan, Z. V. Khaidukov, and M. A. Zubkov. Physical Review D 98.7 (2018): 076013.*
- *C. Banerjee, M. Lewkowicz, M.A. Zubkov, Physics Letters B, 136457 (2021)*
- *C Banerjee, IV Fialkovsky, M Lewkowicz, CX Zhang, MA Zubkov Journal of Computational Electronics 20, 2255-2283 (2021)*
- *C. Banerjee, M. Lewkowicz, M.A. Zubkov, Physical Review D 106 (7), 074508 (2022)*
- *M.A. Zubkov (2023) Journal of Physics A 56 (39), 395201*
- *I.V. Fialkovsky, M.A. Zubkov (2020) Nuclear Physics B 954, 114999*
- *R. Chobanyan, M.A. Zubkov Symmetry 2024, 16(8), 1081*
- *Xavier, P.D., M.A.Zubkov. Physical Review D 112.5 (2025): 056035.*
- *P.D.Xavier, M.A.Zubkov, Physics Letters B, 2025, 140021, https://doi.org/10.1016/j.physletb.2025.140021.*

What is non – dissipative transport?
(CME,CSE,CVE,QHE, …)

Appearance of current (electric, axial, energy) that flows without dissipation.

The conductivities of all known non – dissipative transport phenomena are given by topological invariants.

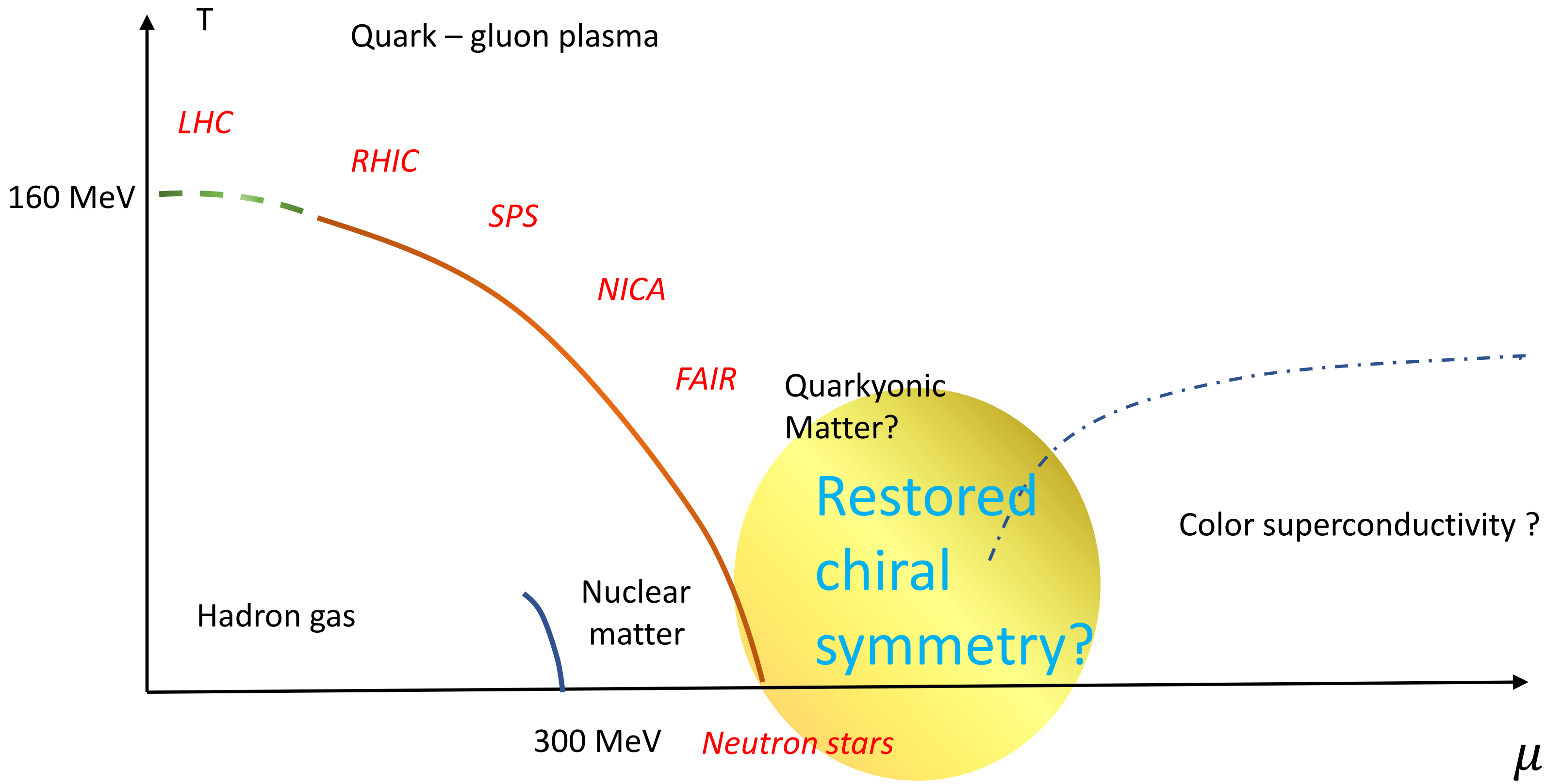
Non – dissipative transport in quark matter

Chiral separation effect (CSE): Axial current in the presence of magnetic field

Chiral vortical effect (CVE): Axial current in the presence of rotation

Chiral magnetic effect (CME): Vector current in the presence of magnetic field

And chiral disbalance



Non – dissipative transport in condensed matter

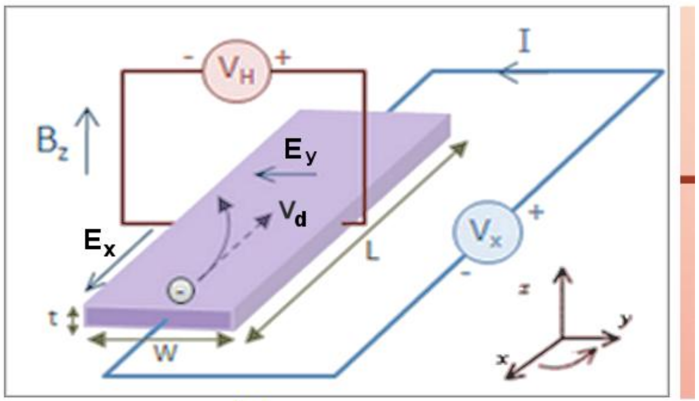
Quantum Hall effect (QHE): Electric current orthogonal to electric field

Chiral separation effect (CSE): Axial current in the presence of magnetic field

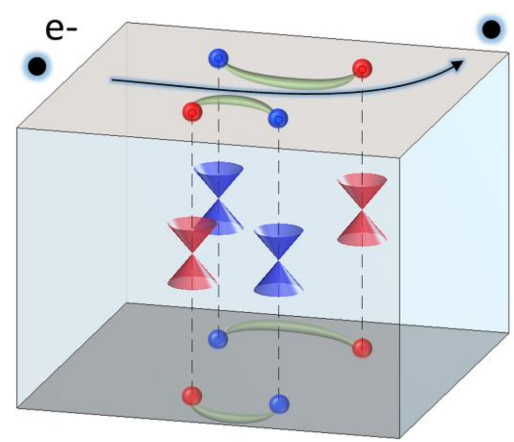
Chiral vortical effect (CVE): Axial current in the presence of rotation

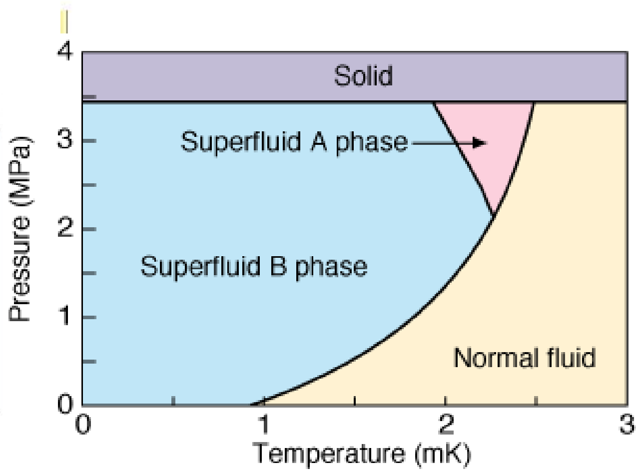
Chiral magnetic effect (CME): Vector current in the presence of magnetic field

And chiral disbalance



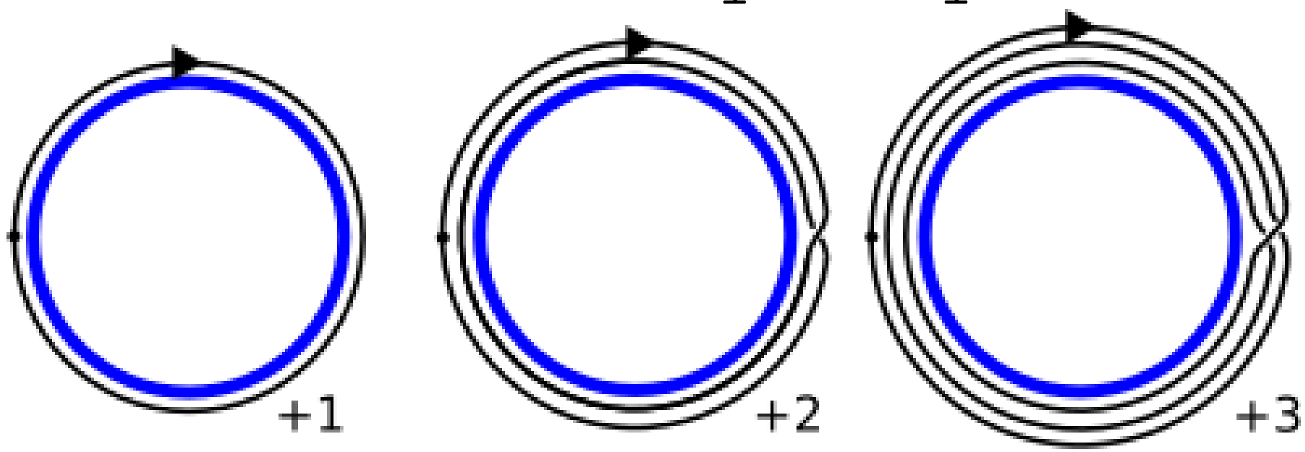
(a) Hall Effect





2d materials: QHE 3d Weyl semimetals: CSE, CME, QHE He3-A superfluid:CVE

Degree of mapping $S_1 \rightarrow S_1$



The first circle winds n times (1,2,3) around the second circle.

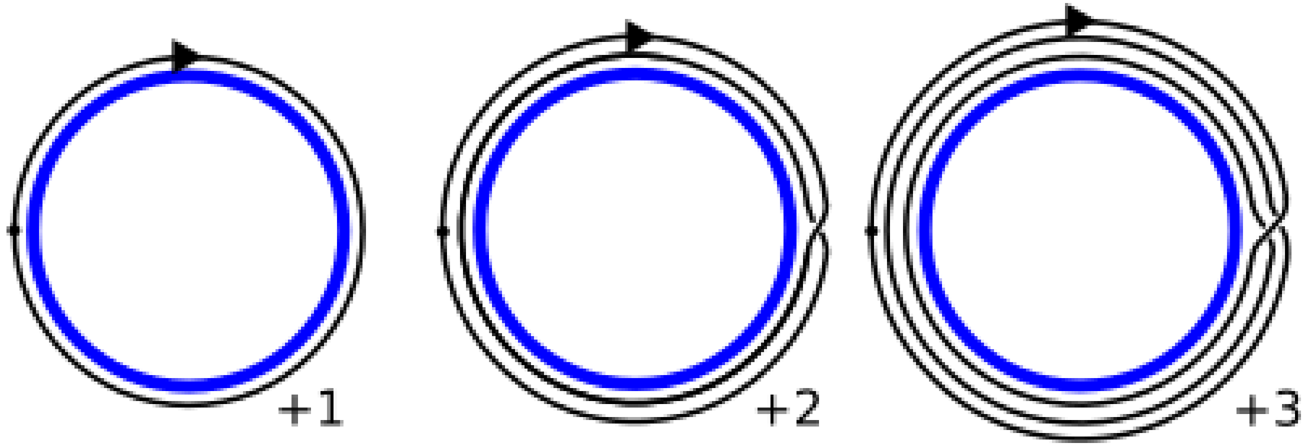
In complex plane this mapping is given by function

$Q(z) = z^n : S1 \rightarrow S1$,

where the first circle is $z(\phi)=r\, e^{i\phi}$, $\phi \in [0,2\pi)$

$$degree[Q] = \frac{1}{2\pi i}\int_0^{2\pi} Q^{-1}(z(\varphi))dQ(z(\varphi)) = n$$

Degree of mapping $S_1 \rightarrow U(N)$



The first circle winds n times (1,2,3) around the second circle.

This mapping is given by function

$Q(z) = e^{in\phi} : S1 \rightarrow U(N)$,

where the circle is $z(\phi)=r\, e^{i\phi}$, $\phi \in [0,2\pi)$

$$degree[Q] = \frac{1}{2\pi i N}\int_0^{2\,\pi} Tr\, Q^{-1}(z(\varphi))dQ(z(\varphi)) = \text{n}$$

Degree of mapping S_3 ➔ SU(N)

S_3 winds around U(N) n times $(\pi_3(SU(N)) = Z)$

Q: S_3 ➔ SU(N)

$$degree[Q] = \frac{1}{24\pi^2}\int_{S_3} Tr\, Q^{-1}dQ \wedge Q^{-1}dQ \wedge Q^{-1}dQ = n$$

This is topological invariant: it is not changed if function Q is changed smoothly

Intrinsic Anomalous Quantum Hall Effect

QHE

homogeneous system

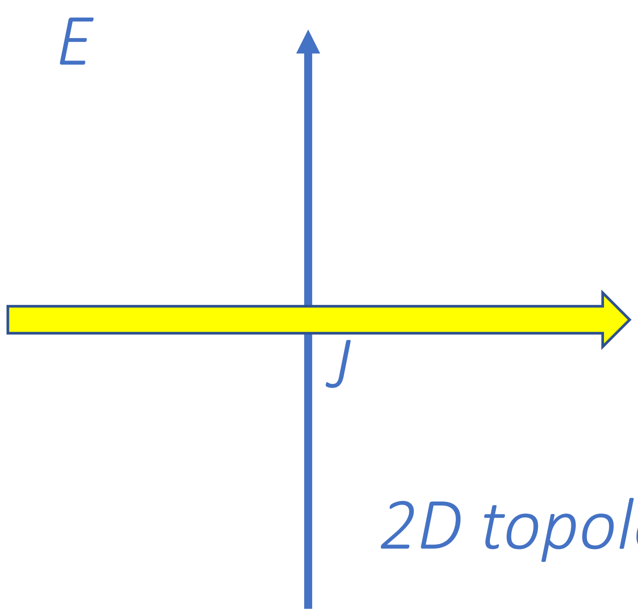
no magnetic field

no interactions

no disorder

T. Matsuyama, Quantization of Conductivity Induced by Topological Structure of Energy Momentum Space in Generalized QED in Three-dimensions, Prog. Theor. Phys 77, 711 (1987)

$$\mathcal{N} = \frac{\epsilon_{ijk}}{3!\,4\pi^2} \int d^3p \, \mathrm{Tr} \left[G(p) \frac{\partial G^{-1}(p)}{\partial p_i} \frac{\partial G(p)}{\partial p_j} \frac{\partial G^{-1}(p)}{\partial p_k} \right]$$



$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

2D topological insulator (Chern insulator)

Intrinsic Anomalous Quantum Hall Effect QHE

homogeneous system

no magnetic field

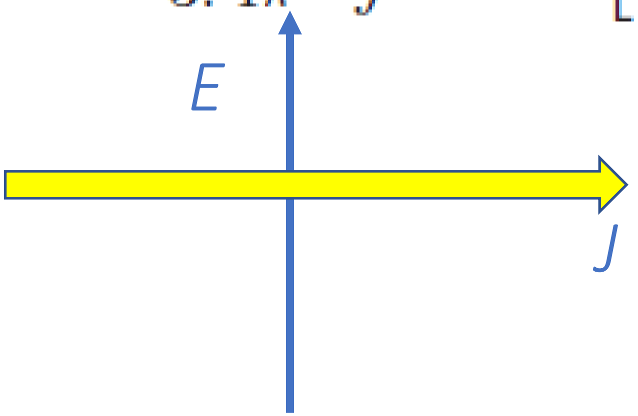
with interactions

no disorder

NOT RENORMALIZED BY INTERACTIONS

2D topological insulator (Chern insulator)

$$\mathcal{N} = \frac{\epsilon_{ijk}}{3!\,4\pi^2}\int d^3p\,\mathrm{Tr}\left[G(p)\frac{\partial G^{-1}(p)}{\partial p_i}\frac{\partial G(p)}{\partial p_j}\frac{\partial G^{-1}(p)}{\partial p_k}\right]$$



$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

Influence of interactions on the anomalous quantum Hall effect

C.X. Zhang, M.A. Zubkov

Journal of Physics A: Mathematical and Theoretical 53 (19), 195002 (2020)

Intrinsic Anomalous Quantum Hall Effect QHE

homogeneous system

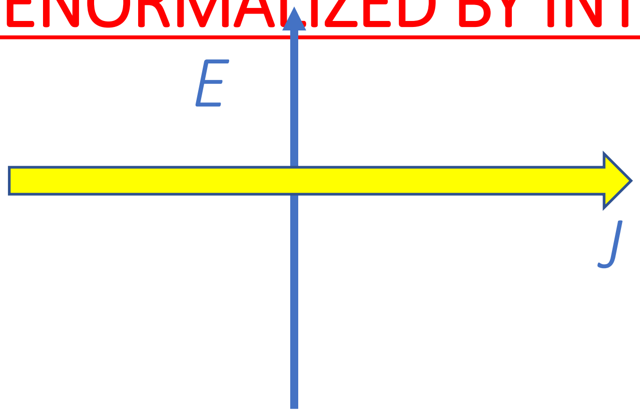
no magnetic field

with interactions

no disorder

the particular case of massive relativistic 2D fermions intereacting with 2D U(1) gauge field

NOT RENORMALIZED BY INTERACTIONS



$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

Coleman S. and Hill B. 1985 Phys. Lett. B159 184. Lee T 1986 Phys. Lett. B171, 247.

Applications to Quantum Hall Effect

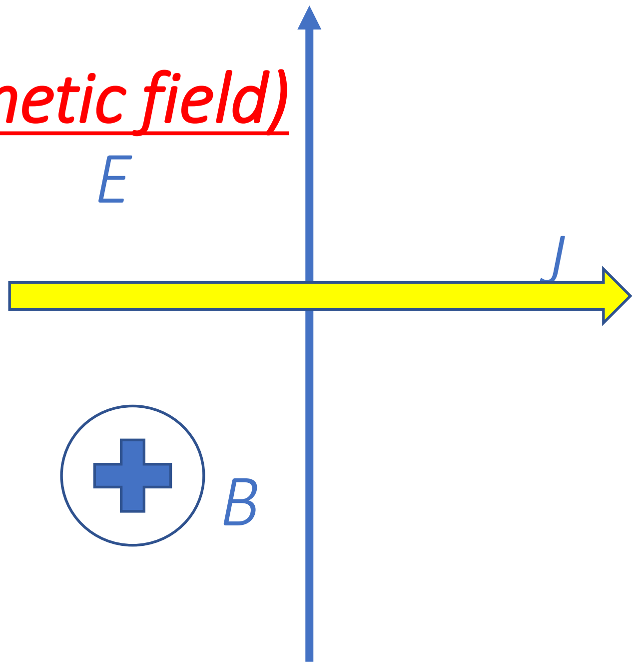
QHE

Equilibrium, T=0

non-homogeneous system

(in particular, in the presence of magnetic field)

Average electric current



$$\langle j^k \rangle = -\frac{1}{2\pi} \mathcal{N} \epsilon^{3kj} E_j,$$

2+1 D:

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\, 3!\, 4\pi^2} \int d^3p d^3x \, \mathrm{Tr} \left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k} \right]$$

M.A. Zubkov *,1, Xi Wu

Annals of Physics 418 (2020) 168179

Quantum Hall Effect ***Equilibrium, T=0*** **QHE**

non-homogeneous system

Average electric current

2+1 D:

$$\langle j^k \rangle = -\frac{1}{2\pi} \mathcal{N} \epsilon^{3kj} E_j,$$

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\, 3!\, 4\pi^2} \int d^3p d^3x \, \mathrm{Tr} \left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k} \right]$$

smooth deformation of the system

↓

the system without disorder, elastic deformations etc, with constant magnetic field

N is not changed!

If N is known for less complicated system, we know it also for the more complicated one

The absence of (perturbative) interaction corrections to Quantum Hall Effect

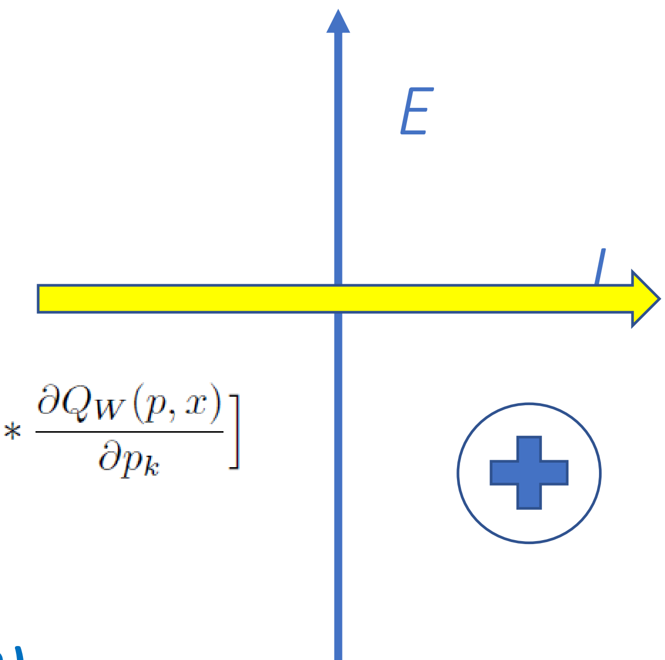
QHE

equilibrium, T=0

Electric current orthogonal to electric field in the presence of magnetic field

$$\langle j^k \rangle = -\frac{1}{2\pi}\mathcal{N}\epsilon^{3kj}E_j,$$

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\,3!\,4\pi^2}\int d^3p d^3x \,\mathrm{Tr}\left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}\right]$$



C.X. Zhang, M.A. Zubkov
Annals of Physics 444, 169016 (2022)

Relation to cyclic cohomology theory

QHE

Topological invariant in phase space

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\,3!\,4\pi^2}\int d^3p d^3x \,\mathrm{Tr}\left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}\right]$$

*One can consider the algebra of functions G_W on phase space with the Moyal product as a product. Then Q_W is inverse to G_W. Let us omit subscript W and *, denote Q_W as G^{-1} and* $\int d^3x\, d^3p$ *Tr as* **Tr**:

$$\mathcal{N} = -\frac{T\,\varepsilon_{ijk}}{3!4\pi^2\,S}\,\mathbf{Tr}\left[G\,\frac{\partial G^{-1}}{\partial p_i}\frac{\partial G}{\partial p_j}\frac{\partial G^{-1}}{\partial p_k}\right]$$

Relation to cyclic cohomology theory

QHE

Here are the alternative notations:

the topological invariant in phase space

$$\mathcal{N} = -\frac{T\,\varepsilon_{ijk}}{3!4\pi^2\,S}\,\mathbf{Tr}\left[G\,\frac{\partial G^{-1}}{\partial p_i}\frac{\partial G}{\partial p_j}\frac{\partial G^{-1}}{\partial p_k}\right]$$

function G produces a K – theory class [G]

$Ch_1\,([G])$ *is the Chern – Connes character of [G] (element of cyclic cohomology).*

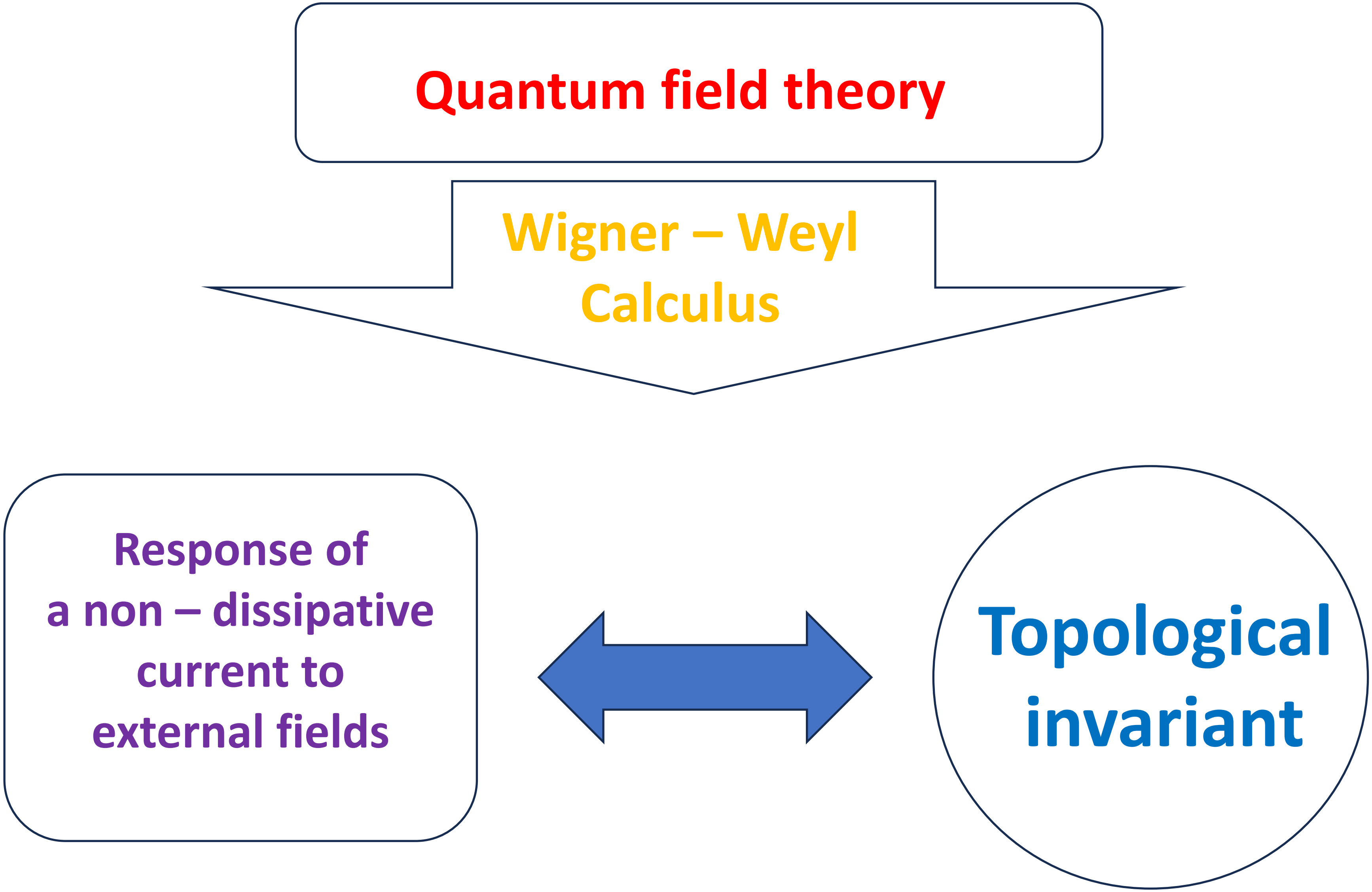
cyclic 3-cocycle $\tau_3(a_0,a_1,a_2,a_3) = 1/(3!) \int Tr(a_0 \partial_{pi} a_1 \partial_{pj} a_2 \partial_{pk} a_3)\varepsilon^{ijk} d^3x d^3p$

It corresponds to cyclic homology class $[\tau_3]$

Now

$\mathcal{N} = \langle Ch_1([G]), [\tau_3]\rangle T/S$ is pairing of elements of cyclic homology and cohomology

Non – dissipative transport phenomena vs. topological invariants



We extend the consideration to the non – Abelian versions of the chiral separation effect and quantum Hall effect.

Xavier, Praveen D., and M. A. Zubkov. "Generalized Wigner-Weyl calculus for gauge theory and nondissipative transport." Physical Review D 112.5 (2025): 056035.

We also would like to obtain expression for chiral anomaly in the presence of external non - Abelian gauge field in the case when topology of fermions in momentum space is nontrivial.

Praveen D. Xavier, M.A. Zubkov, Chiral anomaly in inhomogeneous systems with nontrivial momentum space topology, Physics Letters B, 2025, 140021, ISSN 0370-2693, https://doi.org/10.1016/j.physletb.2025.140021.

Conventional Wigner – Weyl calculus model with fermions | Covariant Wigner – Weyl calculus model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action | typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

$$\hat{Q}(\partial_x) = i\gamma^\mu \partial_\mu - M$$

$$Q = \sum_{|\alpha|\leq m} c_\alpha(x)(-iD)^\alpha$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \; |\alpha| := \sum_\mu \alpha_\mu$$

$$(-i\partial)^\alpha := \prod_\mu (-i\partial_\mu)^{\alpha_\mu}$$

Green function | Green function

$$\hat{G} := \hat{Q}^{-1}$$

Euclidean space - time

conventional Wigner – Weyl calculus

Weyl symbol of operator

$$A_W(x,p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy} \langle x+\frac{y}{2}|\,\hat{A}\,|x-\frac{y}{2}\rangle = \int_{-\infty}^{\infty} dq e^{iqx} \langle p+\frac{q}{2}|\,\hat{A}\,|p-\frac{q}{2}\rangle$$

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covariant Wigner – Weyl calculus

Weyl symbol of operator

$$X_W(x,p) := \int d^4y\, e^{ipy}\, U(x,x-y/2)\, \langle x-y/2|\,\hat{X}\,|x+y/2\rangle\, U(x+y/2,x)$$

$$U(y,x) = \mathrm{Pexp}\left(i \int_{x\to y} dz^\mu\, A_\mu(z) \right)$$

$$X_W(x,p) = \int d^4y\, e^{ipy}\, \langle x|\, e^{-\frac{i}{2}y\hat{\pi}}\, \hat{X}\, e^{-\frac{i}{2}y\hat{\pi}}\, |x\rangle$$

where $\hat{\pi}_\mu := \hat{p}_\mu - A_\mu(\hat{x})$

conventional Wigner – Weyl calculus

Moyal product

$$(f \star g)(x,p) := (2\pi)^{-8} \int d^4y d^4k d^4y' d^4k' \, e^{-iy(k-p)-iy'(k'-p)} f(x-y'/2,k) g(x+y/2,k')$$

the product of two operators

$$(AB)_W(x,p) \equiv A_W(x,p) \star B_W(x,p)$$

---

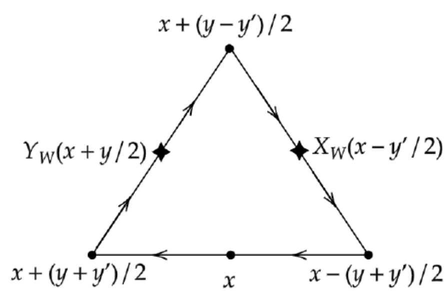
covariant Wigner – Weyl calculus

Star product

$$X_W \bigstar Y_W := (\hat{X}\hat{Y})_W$$

$$(X_W \bigstar Y_W)(x,p)$$
$$= (2\pi)^{-8} \int d^4y d^4k d^4y' d^4k' \, e^{-iy(k-p)-iy'(k'-p)} \times$$
$$U(x, x-(y+y')/2)\, U(x-(y+y')/2, x-y'/2)\, X_W(x-y'/2,k)\, U(x-y'/2, x+(y-y')/2)$$
$$U(x+(y-y')/2, x+y/2)\, Y_W(x+y/2,k')\, U(x+y/2, x+(y+y')/2)\, U(x+(y+y')/2, x)$$

Wilson loop



conventional Wigner – Weyl calculus

Moyal product

$$A_W(x,p) \star B_W(x,p) = A_W(x,p) e^{\overleftrightarrow{\Delta}} B_W(x,p)$$

$$\overleftrightarrow{\Delta} \equiv \frac{i}{2} \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x \right)$$

the product of two operator

$$(AB)_W(x,p) \equiv A_W(x,p) \star B_W(x,p)$$

---

covariant Wigner – Weyl calculus

Moyal product

$$X_W \bigstar Y_W := (\hat{X}\hat{Y})_W$$

$$X_W(x,p) \bigstar Y_W(x,p) =$$

$$\Big(e^{\frac{i}{2}(\overrightarrow{\partial}_{p_1} + \overrightarrow{\partial}_{p_2}) \overrightarrow{D}_x} e^{-\frac{i}{2} \overrightarrow{\partial}_{p_1} \overrightarrow{D}_x} X_W(x,p_1) e^{-\frac{i}{2} \overrightarrow{D}_x \overleftarrow{\partial}_{p_1}}$$

$$e^{-\frac{i}{2} \overrightarrow{\partial}_{p_2} \overrightarrow{D}_x} Y_W(x,p_2) e^{-\frac{i}{2} \overleftarrow{\partial}_{p_2} \overrightarrow{D}_x} e^{\frac{i}{2}(\overleftarrow{\partial}_{p_1} + \overleftarrow{\partial}_{p_2}) \overrightarrow{D}_x} \Big) \times 1 \Big|_{p_1 = p_2 = p}$$

Conventional Wigner – Weyl calculus model with fermions

Covariant Wigner – Weyl calculus model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

$$\hat{Q}(\partial_x) = i\gamma^\mu \partial_\mu - M$$

$$Q = \sum_{|\alpha| \le m} c_\alpha(x)(-iD)^\alpha$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \; |\alpha| := \sum_\mu \alpha_\mu$$

$$(-i\partial)^\alpha := \prod_\mu (-i\partial_\mu)^{\alpha_\mu}$$

Green function

Green function

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

$$G_W(x,p) \bigstar Q(x,p) = 1$$

Conventional Wigner – Weyl calculus model with fermions

Covariant Wigner – Weyl calculus model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x) \hat{Q}(\partial_x) \psi(x)$$

Green function

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

Green function

$$G_W(x,p) \bigstar Q(x,p) = 1$$

$$Q_W(x,p) = \int d^4y \, e^{ipy} \, \langle x - y/2 | \, \hat{Q}^{(A=0)} \, | x + y/2 \rangle$$

$$Q_W(x,p) \equiv (\hat{Q})_W(x,p) = Q(x,p)$$

$$Q(x,p) = \sum_{|\alpha| \le m} o_\alpha(x) p^\alpha$$

Conventional Wigner – Weyl calculus
Green function

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

$$G^{(1)}_{0,W}(x,p) = -\frac{\partial G^{(0)}_{0,W}}{\partial p_\mu}\delta A_\mu - \frac{i}{2} G^{(0)}_{0,W} \star \frac{\partial Q_W}{\partial p_\mu} \star \frac{\partial G^{(0)}_{0,W}}{\partial p_\nu}\delta F_{\mu\nu}$$

Covariant Wigner – Weyl calculus

$$G_W(x,p)\bigstar Q(x,p) = 1$$

$$Q(x,-iD) = \sum_{|\alpha|\le m} o_\alpha(x)\circ(-iD)^\alpha$$

$$o_\alpha(x)\circ(-iD)^\alpha = \frac{1}{2^{|\alpha|}}\{...\{o_\alpha(x),(-iD_1)\}...(-iD_1)\}(-iD_2)\}...(-iD_2)\}...(-iD_4)\}$$

$$Q(x,p) = \sum_{|\alpha|\le m} o_\alpha(x)p^\alpha$$

Conventional Wigner – Weyl calculus

Green function

Covariant Wigner – Weyl calculus

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

$$G_W(x,p) \bigstar Q(x,p) = 1$$

$$Q(x,p) = \sum_{|\alpha| \leq m} o_\alpha(x) p^\alpha$$

$$G_W(x,p,z) = \sum_{n \geq 0} G^{(n)}(x,p,z)$$

$G^{(n)}(x,p,z)$ contains n powers of D_z.

$$G^{(2)}(x,p,z) = -\frac{i}{2} G^{(0)}(x,p) \star \partial_{p_\mu} Q(x,p) \star \partial_{p_\nu} G^{(0)}(x,p) F_{\mu\nu}(z)$$

QUANTUM HALL EFFECT

Conventional QHE (normal Wigner –Weyl calculus)

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

Non – Abelian QHE (Covariant Wigner – Weyl)

$$G_W(x,p)\bigstar Q(x,p) = 1$$

$$G^{(2)}(x,p,z) = -\frac{i}{2} G^{(0)}(x,p) \star \partial_{p_\mu} Q(x,p) \star \partial_{p_\nu} G^{(0)}(x,p) F_{\mu\nu}(z)$$

Abelian Vector current

$$j_k(x) = \frac{\delta \log Z}{\delta A_k(x)}$$

Non – Abelian vector current

$$\langle J_\mu(x)\rangle = -\mathrm{tr}_D \int \frac{d^4p}{(2\pi)^4} G_W \partial_{p_\mu} Q$$

Response to (chromo) Electric field in 2+1 D

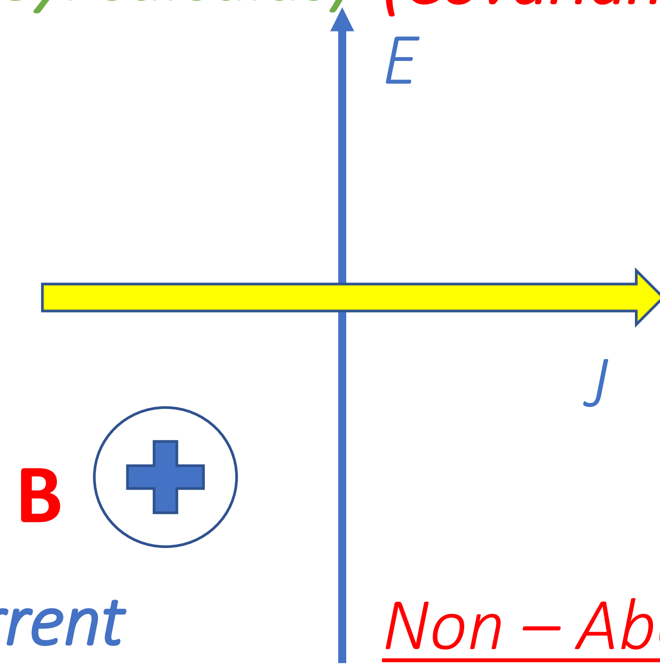
$$\bar{J}_i^{v,QHE} = \frac{1}{2\pi}\epsilon_{ij} M_3 E_j$$

$$M_3 = -\frac{1}{S\, 24\pi^2}\Big[\int d^2x \int \mathrm{tr}_D \left(G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ\right)\Big]_{reg}$$

QUANTUM HALL EFFECT

Conventional QHE
(normal Wigner –Weyl calculus)

Non – Abelian QHE
(Covariant Wigner – Weyl)



Abelian Vector current

Non – Abelian vector current

Response to (chromo) Electric field in 2+1 D

$$\bar{J}_i^{v,QHE} = \frac{1}{2\pi}\epsilon_{ij} M_3 E_j$$

$$M_3 = -\frac{1}{S\,24\pi^2}\Big[\int d^2x \int \mathrm{tr}_D \Big(G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ\Big)\Big]_{reg}$$

CHIRAL SEPARATION EFFECT

Conventional QHE (normal Wigner –Weyl calculus) | Non – Abelian QHE (Covariant Wigner – Weyl)

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

$$G_W(x,p)\bigstar Q(x,p) = 1$$

$$G^{(2)}(x,p,z) = -\frac{i}{2} G^{(0)}(x,p) \star \partial_{p_\mu} Q(x,p) \star \partial_{p_\nu} G^{(0)}(x,p) F_{\mu\nu}(z)$$

Abelian axial current | Non – Abelian axial current

$$\langle J_\mu(x) \rangle = -\frac{1}{2} \mathrm{tr}_D \int \frac{d^4p}{(2\pi)^4} G_W \partial_{p_\mu} [Q, \gamma^5]$$

Response to (chromo) Magnetic field and μ in 3+1 D

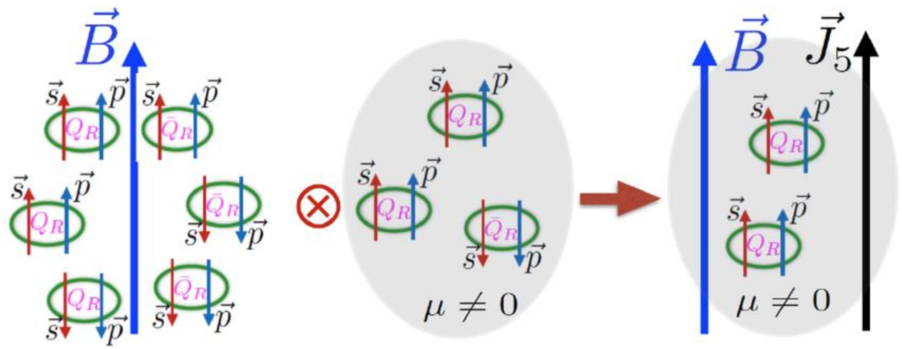
$$\frac{d}{d\mu} \bar{J}_i^{(5)} = \frac{1}{4\pi^2} \epsilon_{ijk} N_3 F_{jk}$$

$$N_3 = -\frac{1}{48\pi^2 V} \int d^3x \int_{\Sigma_0} \mathrm{tr}_D \left(G^{(0)} \star dQ \star \wedge dG^{(0)} \wedge dQ \right)$$

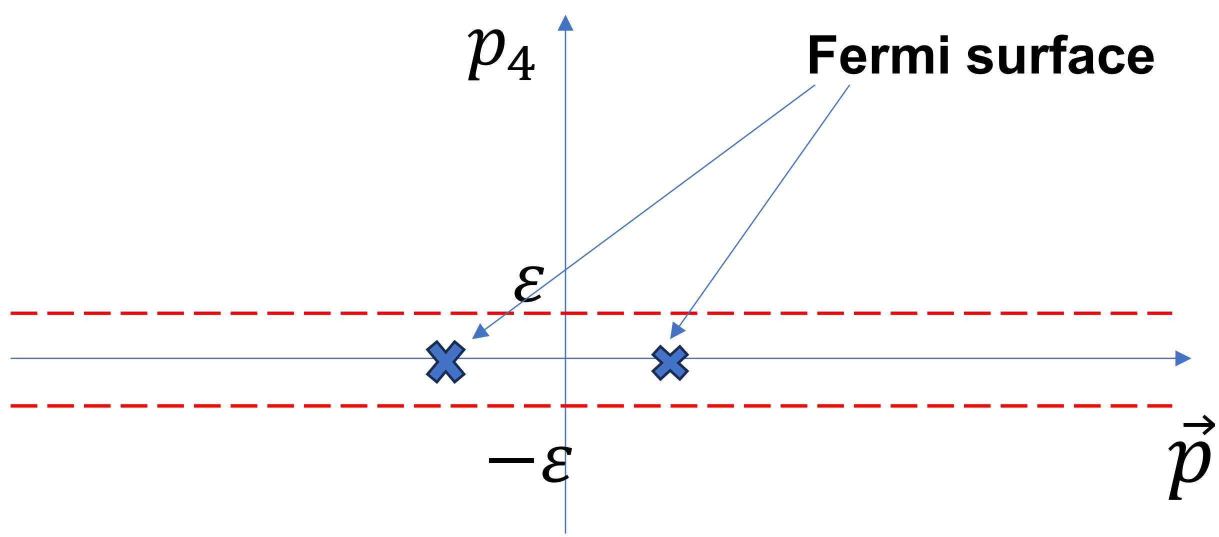
Σ_0 in $4D$ momentum space consists of the two hyperplanes $p_4 = \pm\epsilon \to 0$.

CHIRAL SEPARATION EFFECT

Axial current along magnetic field in the presence of chemical potential



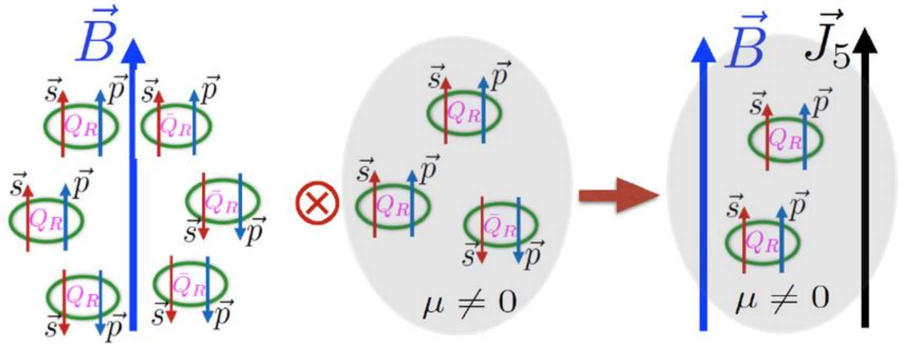
Momentum space



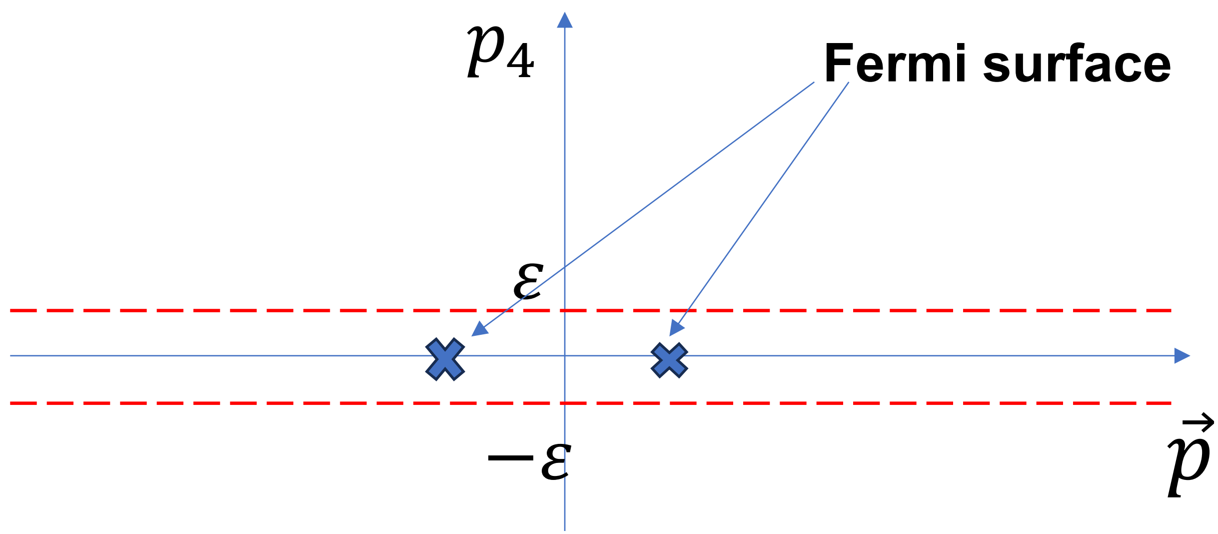
Σ_0 in $4D$ momentum space consists of the two hyperplanes $p_4 = \pm\epsilon \to 0$.

CHIRAL SEPARATION EFFECT

Axial current along magnetic field in the presence of chemical potential



Momentum space



Σ_0 in $4D$ momentum space consists of the two hyperplanes $p_4 = \pm\epsilon \to 0$.

Xavier, Praveen D., and M. A. Zubkov. "Generalized Wigner-Weyl calculus for gauge theory and nondissipative transport." Physical Review D 112.5 (2025): 056035.

Chiral anomaly vs. Atiyah – Singer theorem

$$Z = \int D\bar{\psi} D\psi \, e^{\int d^4x \, \bar{\psi}(x) Q \psi(x)}$$

$$Q = \begin{pmatrix} 0 & O^\dagger \\ O & 0 \end{pmatrix}$$

$$O = \sum_{|\alpha| \leq m} f_\alpha(x)(-i\partial)^\alpha$$

Principal symbol of operator O

$$o(x,p) := \sum_{|\alpha|=m} f_\alpha(x) p^\alpha$$

$$n_+ - n_- = \dim\ker O - \dim\ker O^\dagger = \operatorname{index} O$$

n_+ (resp. n_-) is defined as the number of zero modes of Q with positive (resp. negative) chirality

anomaly

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle = 2i(n_+ - n_-)$$

Atiyah – Singer theorem

$$\operatorname{index} O = \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = \text{topological index } O$$

$$\mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

associated "virtual bundle" ξ

Chiral anomaly vs. Atiyah – Singer theorem

$$Z = \int D\bar{\psi} D\psi \, e^{\int d^4x \, \bar{\psi}(x) Q \psi(x)}$$

$$Q = \begin{pmatrix} 0 & O^\dagger \\ O & 0 \end{pmatrix}$$

$$O = \sum_{|\alpha| \le m} f_\alpha(x)(-i\partial)^\alpha$$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle = 2i(n_+ - n_-)$$

n_+ (resp. n_-) is defined as the number of zero modes of Q with positive (resp. negative) chirality

For the fermions with conventional Dirac operator

$$\mathscr{A} = -\frac{i}{4\pi^2} \int \mathrm{tr} F \wedge F$$

In general case (obtained in our work for the first time)

$$\mathscr{A} = -N_3 \times \frac{i}{4\pi^2} \int \mathrm{tr} F \wedge F$$

$$N_3 := \frac{1}{48\pi^2 |V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

Provided that the topology in coordinate space Is due to the gauge field A only

Σ defined as the union of the two hyperplanes $p_4 = 0^\pm$

$$G^{(0)} \star Q_W = 1$$

Derivation

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$\text{with} \quad S = \int d^4x \, \bar{\psi}(x) Q(x, -iD) \psi(x)$$

$$Q(x, -iD) = \sum_{|\alpha| \leq m} c_\alpha(x) (-iD)^\alpha$$

$$S = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q} \hat{\rho} \right) \qquad \langle x | \, \hat{\rho} \, | y \rangle := \psi(x) \bar{\psi}(y)$$

Regularization: point splitting

$$\hat{\rho}^\epsilon := e^{i\hat{\pi}\epsilon} \hat{\rho} e^{i\hat{\pi}\epsilon}$$

$$\langle x | \, \hat{\rho}^\epsilon \, | x \rangle = U(x, x+\epsilon) \psi(x+\epsilon) \psi(x-\epsilon) U(x-\epsilon, x)$$

$$S^\epsilon = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q} \hat{\rho}^\epsilon \right)$$

Derivation

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$S^{\epsilon} = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q} \hat{\rho}^{\epsilon} \right)$$

$$\langle x | \hat{\rho}^{\epsilon} | x \rangle = U(x, x+\epsilon) \psi(x+\epsilon) \bar{\psi}(x-\epsilon) U(x-\epsilon, x)$$

Noether current corresponding to chiral transformation

$$\psi(x) \to e^{i\alpha(x)\gamma^5} \psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{i\alpha(x)\gamma^5}$$

$$\alpha \in \mathfrak{g}$$

Variation of action

$$\delta S^{\epsilon} = -i \mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\alpha(\hat{x}) \gamma^5 \{ \hat{Q}, \hat{\rho}^{\epsilon} \} \right) \qquad \delta S^{\epsilon} = \mathrm{tr}_G \int d^4 x \, \alpha(x) \Gamma^{\epsilon}(x)$$

$$\Gamma^{\epsilon}(x) = -i \mathrm{tr}_D \gamma^5 \int \frac{d^4 p}{(2\pi)^4} (Q_W \star \rho^{\epsilon}_W + \rho^{\epsilon}_W \star Q_W) \qquad \rho^{\epsilon}_W = e^{ip\epsilon} \star \rho_W \star e^{ip\epsilon}$$

$\Gamma^{\epsilon}(x) = \mathcal{D}_{\mu} J^{\epsilon}_{\mu}(x)$ axial current: higher orders in derivatives

$$J^{\epsilon}_{\mu}(x) := -\frac{1}{2} \mathrm{tr}_D \gamma^5 \int \frac{d^4 p}{(2\pi)^4} \left(\partial_{p_\mu} Q_W(x,p) \rho^{\epsilon}_W(x,p) - \rho^{\epsilon}_W(x,p) \partial_{p_\mu} Q_W(x,p) \right) + \ldots$$

Derivation

$$Z = \int D\bar{\psi} D\psi \, e^S$$

$$S^\epsilon = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q} \hat{\rho}^\epsilon \right)$$

$$\psi(x) \to e^{i\alpha(x)\gamma^5} \psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{i\alpha(x)\gamma^5}$$

$$\delta S^\epsilon = \mathrm{tr}_G \int d^4x \, \alpha(x) \Gamma^\epsilon(x) \qquad \Gamma^\epsilon(x) = \mathcal{D}_\mu J^\epsilon_\mu(x)$$

axial current: higher orders in covariant derivatives

$$J^\epsilon_\mu(x) := -\frac{1}{2} \mathrm{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} \left(\partial_{p_\mu} Q_W(x,p) \rho^\epsilon_W(x,p) - \rho^\epsilon_W(x,p) \partial_{p_\mu} Q_W(x,p) \right) + \ldots$$

Chiral anomaly:

$$\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = i \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4p \left(Q_W \bigstar e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} + e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} \bigstar Q_W \right)$$

With extra integration over x we have a divergent expression → infrared regularization (integration over a finite region of space)

Expansion in powers of F: sum of $\sim e^{2i\epsilon p} \epsilon^n F^m$ with $m \geq n$

The terms with n> 1 are irrelevant in the limit $\epsilon \to 0$

$$\int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = -2i \, \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_\mu \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

$$Z = \int D\bar{\psi} D\psi \, e^S$$

Chiral anomaly:

$$\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = i \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4p \left(Q_W \bigstar e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} + e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} \bigstar Q_W \right)$$

Up to the terms, which do not disappear in the limit $\epsilon \to 0$

$$\int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = -2i \, \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_\mu \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

We use the theorem (averaging over directions)

$$\lim_{|\epsilon| \to 0} \left\langle \int d^4p \, e^{ip\epsilon} \epsilon_\mu f(p) \right\rangle = i \int d^4p \, \partial_\mu f(p)$$

$$\lim_{|\epsilon| \to 0} \int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = + \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, \partial_{p_\mu} \left(\left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W \right)$$

Topology in coordinate space is due to the gauge field only

$Q_W(x,p)$ is homotopic to a function $\tilde{Q}(p)$ $\qquad \tilde{G}_W \bigstar \tilde{Q} = 1$

$$\mathscr{A} = -\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, \partial_{p_\mu} \left((F_{\mu\nu} \partial_{p_\nu} \tilde{Q} - \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} \tilde{Q}) \tilde{G}_W \right)$$

$$\tilde{G}_W = \tilde{G}^{(0)} + \frac{i}{2} \tilde{G}^{(0)} \partial_{p_\alpha} \tilde{Q} \, \tilde{G}^{(0)} \partial_{p_\beta} \tilde{Q} \, \tilde{G}^{(0)} F_{\alpha\beta} + O(F^2)$$

Chiral anomaly:

$$\lim_{|\epsilon|\to 0}\int d^4x\, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = +\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p\, \partial_{p_\mu} \Big(\Big(\partial_{x_\mu} Q_W - F_{\mu\nu}\partial_{p_\nu} Q_W + \frac{1}{24}\mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha}\partial_{p_\beta}\partial_{p_\nu} Q_W \Big) G_W \Big)$$

Topology in coordinate space is due to the gauge field only

$Q_W(x,p)$ is homotopic to a function $\widetilde{Q}(p)$

$$\widetilde{G}_W \star \widetilde{Q} = 1$$

$$\mathscr{A} = -\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p\, \partial_{p_\mu} \Big((F_{\mu\nu}\partial_{p_\nu}\widetilde{Q} - \frac{1}{24}\mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha}\partial_{p_\beta}\partial_{p_\nu}\widetilde{Q})\widetilde{G}_W \Big)$$

$$\widetilde{G}_W = \widetilde{G}^{(0)} + \frac{i}{2}\widetilde{G}^{(0)}\partial_{p_\alpha}\widetilde{Q}\,\widetilde{G}^{(0)}\partial_{p_\beta}\widetilde{Q}\,\widetilde{G}^{(0)} F_{\alpha\beta} + O(F^2)$$

$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x\, \mathrm{tr}(FF^\star)$$

$$N_3 = \frac{1}{8\pi^2}\int dS$$

$$S_{\alpha\beta\nu}(x) := \frac{1}{2}\mathrm{tr}_D \Big(\gamma^5 \widetilde{G}^{(0)}\partial_{p_\alpha}\widetilde{Q}\,\widetilde{G}^{(0)}\partial_{p_\beta}\widetilde{Q}\,\widetilde{G}^{(0)}\partial_{p_\nu}\widetilde{Q} \Big) - (\alpha \leftrightarrow \beta)$$

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

Chiral anomaly:

$$\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = i \,\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4p \left(Q_W \bigstar e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} + e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} \bigstar Q_W \right)$$

Up to the terms, which do not disappear in the limit $\epsilon \to 0$

$$\int d^4x \,\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = -2i \,\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_\mu \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

We use the theorem (averaging over directions)

$$\lim_{|\epsilon|\to 0} \left\langle \int d^4p \, e^{ip\epsilon} \epsilon_\mu f(p) \right\rangle = i \int d^4p \, \partial_\mu f(p)$$

$$\lim_{|\epsilon|\to 0} \int d^4x \,\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = +\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, \partial_{p_\mu} \left(\left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W \right)$$

Topology in coordinate space is due to the gauge field only

$Q_W(x,p)$ is homotopic to a function $\widetilde{Q}(p)$

$$\mathscr{A} = -2i N_3 \int \frac{1}{16\pi^2} d^4x \,\mathrm{tr}(F F^\star)$$

$$N_3 = \frac{1}{8\pi^2} \int_\Sigma S = \frac{1}{48\pi^2} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 \widetilde{G}^{(0)} d\widetilde{Q} \, \widetilde{G}^{(0)} \wedge d\widetilde{Q} \widetilde{G}^{(0)} \wedge d\widetilde{Q} \right)$$

Chiral anomaly:

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = i \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4 p \left(Q_W \bigstar e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} + e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} \bigstar Q_W \right)$$

Up to the terms, which do not disappear in the limit $\epsilon \to 0$

$$\int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = -2i \, \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_\mu \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

We use the theorem (averaging over directions)

$$\lim_{|\epsilon| \to 0} \left\langle \int d^4p \, e^{ip\epsilon} \epsilon_\mu f(p) \right\rangle = i \int d^4p \, \partial_\mu f(p)$$

$$\lim_{|\epsilon| \to 0} \int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = + \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, \partial_{p_\mu} \left(\left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W \right)$$

Topology in coordinate space is due to the gauge field only

$Q_W(x,p)$ is homotopic to a function $\widetilde{Q}(p)$

$$\mathscr{A} = -2i N_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(F F^\star)$$

$$N_3 = \frac{1}{48\pi^2 |V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

Chiral anomaly:

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

Topology in coordinate space is due to the gauge field only

⇕

$Q_W(x,p)$ is homotopic to a function $\widetilde{Q}(p)$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle$$

$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(F F^\star)$$

$$N_3 = \frac{1}{48\pi^2|V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

In Minkowski space – time:

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$

Chiral anomaly:

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

Topology in coordinate space is due to the gauge field only

$\Updownarrow$

$$Q_W(x,p) \text{ is homotopic to a function } \widetilde{Q}(p)$$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle \qquad \mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(FF^\star)$$

$$N_3 = \frac{1}{48\pi^2|V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

In Minkowski space – time:

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$

In Minkowski space – time:

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle \qquad \mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(FF^\star)$$

$$N_3 = \frac{1}{48\pi^2|V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$

Praveen D. Xavier, M.A. Zubkov, Chiral anomaly in inhomogeneous systems with nontrivial momentum space topology, Physics Letters B, 2025, 140021, ISSN 0370-2693, https://doi.org/10.1016/j.physletb.2025.140021.

N_3 is expressed through Green function, which means it is valid for the interacting case (conjecture)

In Minkowski space – time:

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle \qquad \mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(FF^\star)$$

$$N_3 = \frac{1}{48\pi^2|V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$

The similar result has been obtained in

Dantas, Renato MA, Francisco Peña-Benitez, Bitan Roy, and Piotr Surówka. "Non-Abelian anomalies in multi-Weyl semimetals." *Physical Review Research* 2, no. 1 (2020): 013007.

(but N_3 was expressed there through Berry curvature, which means That unlike our expression it is not valid for the interacting case)

Example

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$\hat{Q} = \begin{pmatrix} 0 & \hat{O}^{\dagger} \\ \hat{O} & 0 \end{pmatrix}$$

$$\hat{O} = \hat{\pi}_4 + i \begin{pmatrix} \hat{\pi}_3 & \kappa(\hat{\pi}_1 - i\hat{\pi}_2)^n \\ \kappa(\hat{\pi}_1 + i\hat{\pi}_2)^n & -\hat{\pi}_3 \end{pmatrix}$$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle \qquad \mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

Topology in coordinate space is due to the gauge field only

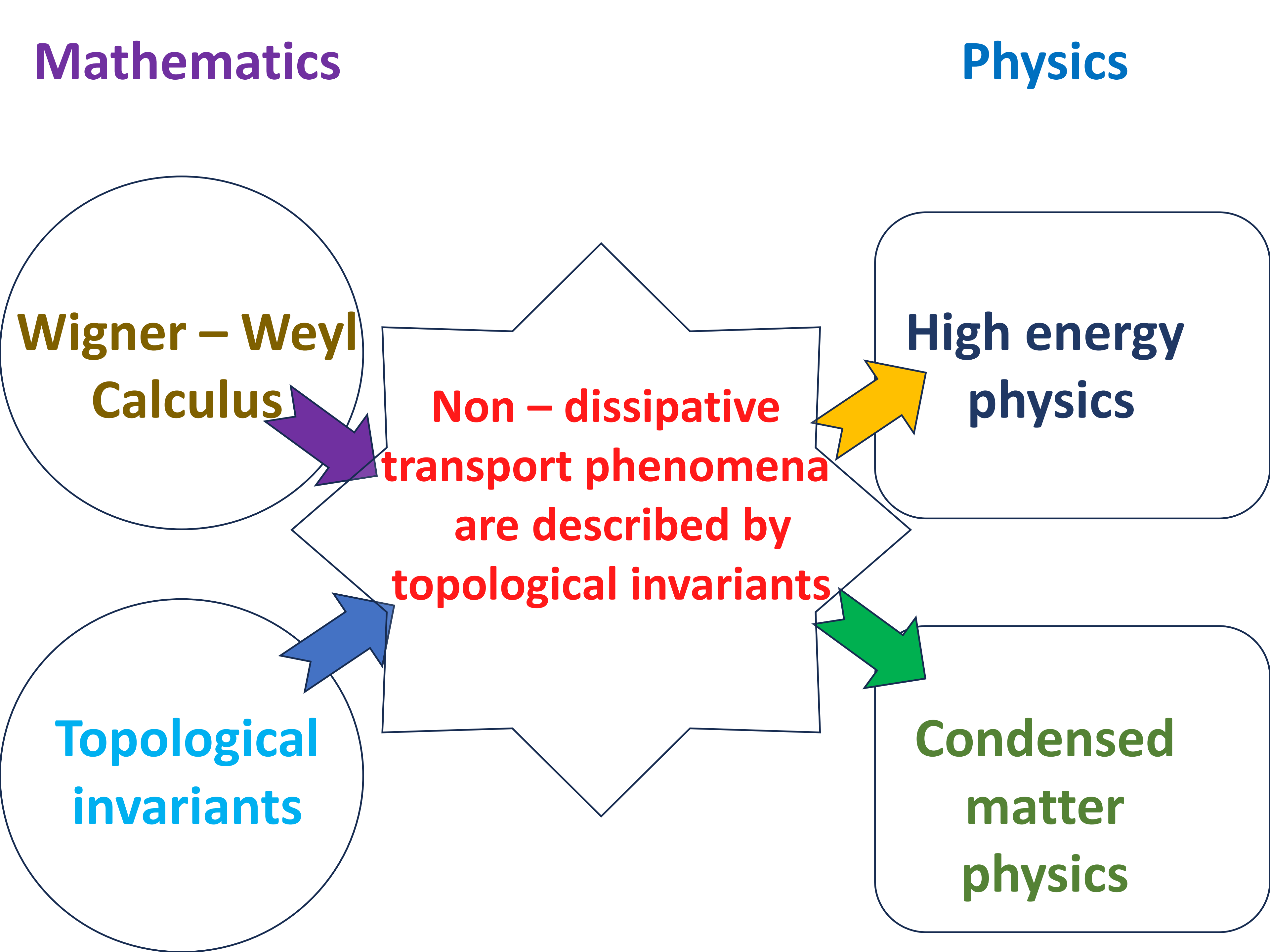
$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(FF^{\star})$$

$$N_3 = n$$

$$N_3 = \frac{1}{48\pi^2 |V|} \int d^3\vec{x} \int_{\Sigma} \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

In Minkowski space – time:

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$


Mathematics
Physics
Wigner – Weyl Calculus
Topological invariants
Non – dissipative transport phenomena are described by topological invariants
High energy physics
Condensed matter physics

Wigner – Weyl calculus in continuum theory
Equilibrium, T=0

model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

$$\hat{Q}(\partial_x) = i\gamma^\mu \partial_\mu - M$$

Green function

$$(i\gamma_\mu \partial^\mu_x - m)G(x-y) = \delta(x-y)$$

Wigner – Weyl calculus in continuum theory

Weyl symbol of operator

$$A_W(x,p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy} \langle x+\frac{y}{2}|\,\hat{A}\,|x-\frac{y}{2}\rangle = \int_{-\infty}^{\infty} dq e^{iqx} \langle p+\frac{q}{2}|\,\hat{A}\,|p-\frac{q}{2}\rangle$$

Wigner – Weyl calculus in continuum theory

Moyal product

$$A_W(x,p) \star B_W(x,p) = A_W(x,p) e^{\overleftrightarrow{\Delta}} B_W(x,p)$$

$$\overleftrightarrow{\Delta} \equiv \frac{i}{2}\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)$$

Weyl symbol of the product of two operators

$$\boxed{(AB)_W(x,p) \equiv A_W(x,p) \star B_W(x,p)}$$

Wigner – Weyl calculus in continuum theory

model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

$$\hat{Q}\,(\partial_x) = i\gamma^\mu\partial_\mu - M$$

Green function

$$(i\gamma_\mu\partial^\mu_x - m)G(x-y) = \delta(x-y)$$

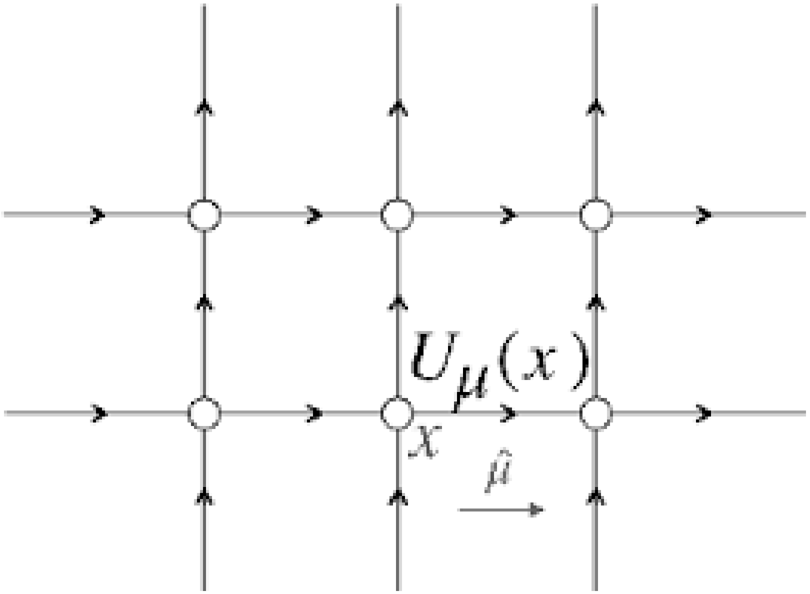
Groenewold equation

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

Lattice models

Example of Wilson fermions

In the presence of gauge field



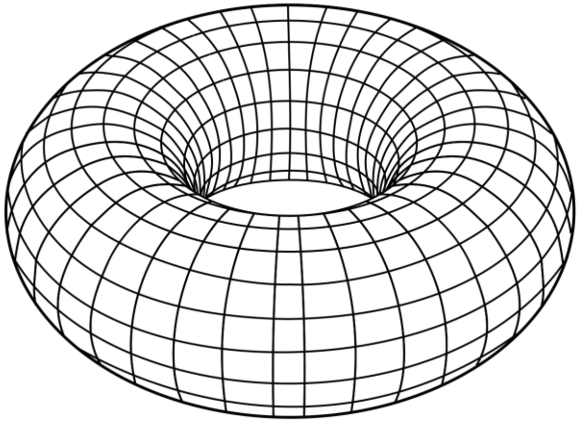
$$S_F^{(W)} = \sum_{\substack{n,m \\ \alpha,\beta}} \hat{\bar{\psi}}_\alpha(n) D_{\alpha\beta}^{(W)}(n,m) \hat{\psi}_\beta(n)$$

$$D_{\mathbf{x},\mathbf{y}} = -\frac{1}{2} \sum_i \left[(1+\gamma^i)\delta_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}} + (1-\gamma^i)\delta_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}} \right] U_{\mathbf{x},\mathbf{y}} + (m^{(0)}+4)\delta_{\mathbf{x},\mathbf{y}}$$

$$U_{x,y} = P e^{i \int_x^y d\boldsymbol{\xi} A(\boldsymbol{\xi})}$$

Approximate *Wigner – Weyl calculus for the lattice models*

Mathematical tools



Weyl symbol of operator (momentum space)

$$[\hat{A}]_W(x_n,p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p+\frac{q}{2}|\,\hat{A}\,|p-\frac{q}{2}\rangle$$

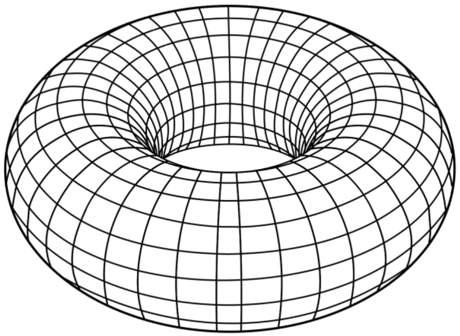
Approximate Wigner – Weyl calculus for the lattice models

Mathematical tools

Weyl symbol of operator (momentum space)

$$[\hat{A}]_W(x_n, p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p + \frac{q}{2}| \hat{A} |p - \frac{q}{2}\rangle$$

Weyl symbol of the product of two operators



This identity is approximate. It is valid for the near diagonal operators

$$(AB)_W(x_n, p) \equiv A_W(x_n, p) \star B_W(x_n, p)$$

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$$(AB)_W(x_n,p) \equiv A_W(x_n,p) \star B_W(x_n,p)$$

It is valid for the near diagonal operators

partition function

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

$$S[\psi,\bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p)$$

Action

Lattice model for the description of electrons in crystals:

In condensed matter systems:

The typical Lattice Dirac operator ***Q*** *is almost diagonal if the external magnetic field strength is much smaller than 10 000 Tesla while wavelength of external electromagnetic field is much larger than 1 nanometer*

This identity is approximate.

$$(AB)_W(x_n,p) \equiv A_W(x_n,p) \star B_W(x_n,p)$$

It is valid for the near diagonal operators

partition function

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

Action

$$S[\psi,\bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p)$$

Lattice model for the regularization of continuum quantum field theory:

The typical Lattice Dirac operator ***Q*** *is almost diagonal when we approach continuum limit of the lattice model.*

Mathematical tools

We can use the approximate Wigner – Weyl calculus dealing with ***any lattice regularized continuum quantum field theory***

and dealing with the lattice models of solid state physics ***if the external magnetic field strength is much smaller than 10 000 Tesla*** *while wavelength of external electromagnetic field is much larger than* ***1 nanometer***

Mathematical tools

partition function

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

Action

$$S[\psi,\bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p)$$

Green function

$$G(p_1,p_2) = \langle p_1 | G | p_2 \rangle =$$

$$\frac{1}{Z} \int D\bar{\psi} D\psi \bar{\psi}(p_2) \psi(p_1) \exp\left(\int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p) \right)$$

Groenewold equation

$$Q_W(p,x) \star G_W(p,x) = 1$$

Moyal product

$$\star_{xp} \equiv e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)}$$

Electric current

$$j_i(x) = \frac{\delta \log Z}{\delta A_k(x)} = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \mathrm{tr}\left[G_W(x,p) \partial_{p_i} Q_W(x,p)\right]$$

Mathematical tools

Groenewold equation

$$Q_W(p,x) \star G_W(p,x) = 1$$

Moyal product

$$\star_{xp} \equiv e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)}$$

Iterative solution

$$G^{(1)}_{0,W}(x,p) = -\frac{\partial G^{(0)}_{0,W}}{\partial p_\mu}\delta A_\mu - \frac{i}{2} G^{(0)}_{0,W} \star \frac{\partial Q_W}{\partial p_\mu} \star \frac{\partial G^{(0)}_{0,W}}{\partial p_\nu}\delta F_{\mu\nu}$$

$$\mathcal{Q}(p - A(R) - \delta A) = \mathcal{Q}(p - A(R)) - \partial^\mu \mathcal{Q} \delta A_\mu$$

Electric current

$$j_i(x) = \frac{\delta \log Z}{\delta A_k(x)} = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr}\left[G_W(x,p)\partial_{p_i} Q_W(x,p)\right]$$

The case of 2D system

$$\langle j^k \rangle = -\frac{1}{2\pi}\mathcal{N}\epsilon^{3kj}E_j,$$

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\, 3!\, 4\pi^2}\int d^3p d^3x \operatorname{Tr}\left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}\right]$$

Mathematical tools

Precise Wigner – Weyl calculus for the lattice models (the details at the end of the talk, if time remains)

Finite rectangular lattice:

M.A. Zubkov (2023)
Journal of Physics A: Mathematical and Theoretical 56 (39), 395201

Infinite rectangular lattice:

I.V. Fialkovsky, M.A. Zubkov (2020)
Nuclear Physics B 954, 114999

Infinite honeycomb lattice:

R. Chobanyan, M.A. Zubkov
arXiv preprint arXiv:2302.00723

We can use the precise Wigner – Weyl calculus dealing with ***any lattice regularized continuum quantum field theory***

and dealing with the lattice models of solid state physics ***if the external magnetic field strength is of the order of 10 000 Tesla (unphysical!)*** *while wavelength of external electromagnetic field is of the order of* ***1 nanometer***

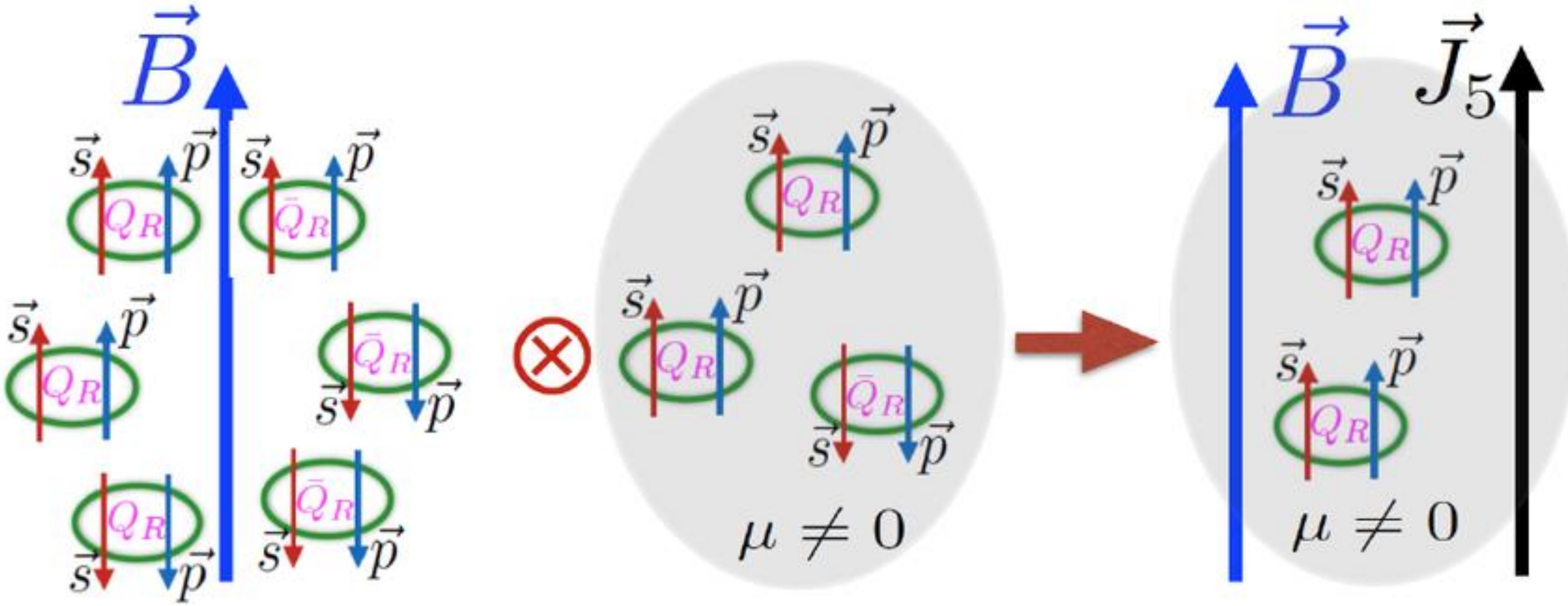
Which is more important, we can use this formalism for artificial lattices, when magnetic flux through the EFFECTIVE lattice cell is compared to 1

And also for the precise treatment of lattice regularized QFT

CHIRAL SEPARATION EFFECT

CSE

Axial current along magnetic field in the presence of chemical potential



D.E. Kharzeev, J. Liao, S.A. Voloshin, G. Wang,
Progress in Particle and Nuclear Physics, Volume 88, 2016, Pages 1-28,

$$J_5^k = -\frac{1}{4\pi^2}\epsilon^{ijk0}\mu F_{ij}$$

A. Metlitski and Ariel R. Zhitnitsky,Phys. Rev. D 72 (2005), 045011

Lattice Dirac operator Q

CSE

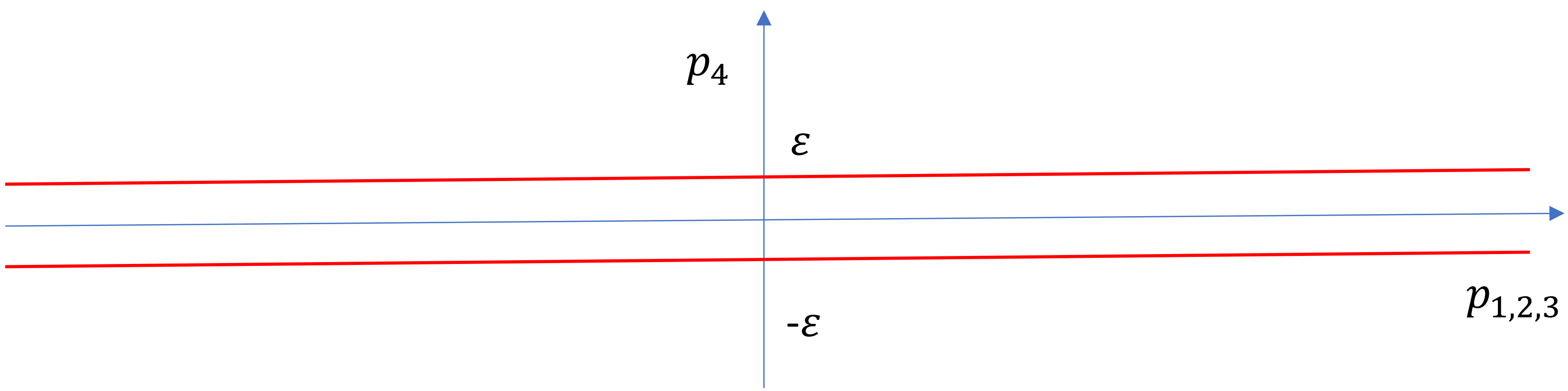
Is 4 x 4 matrix expressed through the Gamma matrices

$$j^5_k(x) = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \mathrm{tr}\,[\gamma^5 G_W(x,p)\partial_{p_k} Q_W(x,p)]$$

The system with Fermi surface of arbitrary complicated form

$$\bar{J}^k_5 = -\frac{\mathcal{N}}{4\pi^2}\epsilon^{ijk0}\mu F_{ij} \quad \mathcal{N} = -\frac{1}{48\pi^2 \mathbf{V}} \int_{\Sigma_3} \int d^3x\, \mathrm{tr}\left[\gamma^5 G_W \star dQ_W \star G_W \wedge \star dQ_W \star G_W \star \wedge dQ_W\right]$$

Surface Σ_3 consists of the two hyperplanes $p_4 = \pm\varepsilon \to 0$



M.Suleymanov, M.Zubkov, Physical Review D 102 (7), 076019 (2020)

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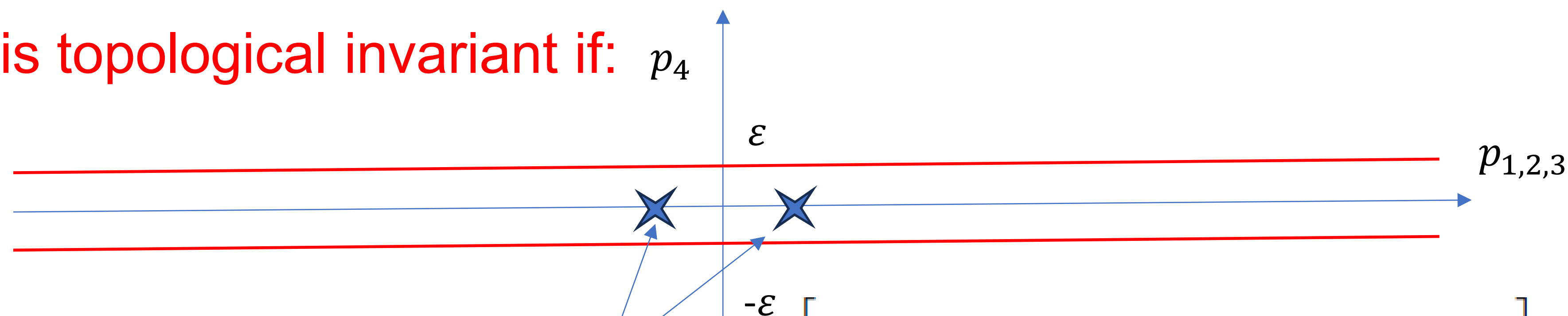
Surface Σ_3 consists of the two hyperplanes $p_4 = \pm \varepsilon \to 0$

N is topological invariant if:



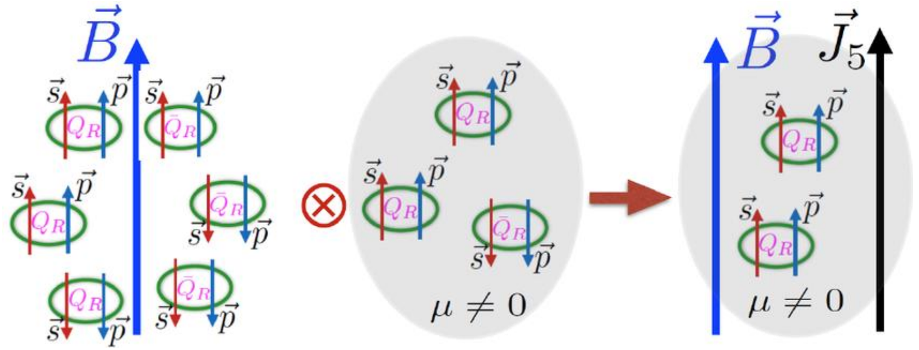
Surface Σ_3 surrounds the singularities of $\left[\gamma^5 G^{(0)}_W \star dQ^{(0)}_W \star G^{(0)}_W \wedge \star dQ^{(0)}_W \star G^{(0)}_W \star \wedge dQ^{(0)}_W \right]$

γ^5 commutes/anticommutes with Q in small vicinity of the singularities

Lattice Dirac operator Q

CSE

is 4 x 4 matrix expressed through the Gamma matrices



The system with Fermi surface of arbitrary complicated form

$$\bar{J}_5^k = -\frac{\mathcal{N}}{4\pi^2}\epsilon^{ijk0}\mu F_{ij}$$

Irrespective of the form of the Fermi surface the value of $\mathcal{N}$ is equal to the number of chiral 4 – component Dirac fermions

M.Suleymanov, M.Zubkov, Physical Review D 102 (7), 076019 (2020)

Non – renormalization of CSE by interactions in QCD

CSE

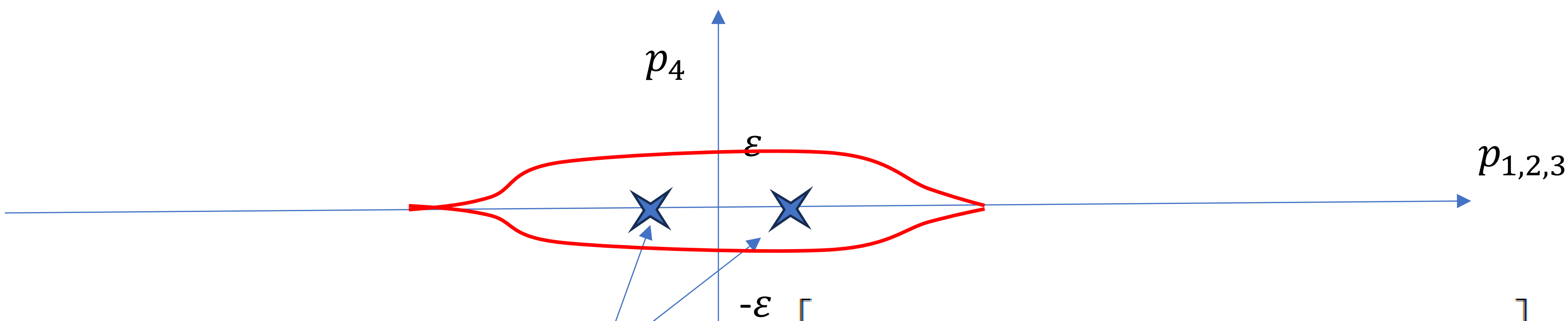
$$\bar{J}_5^k = -\frac{\mathcal{N}}{4\pi^2}\epsilon^{ijk0}\mu F_{ij}$$

Chemical potential is counted from the level, where the CSE disappears (the position of the phase transition)

$$\mathcal{N} = -\frac{1}{48\pi^2 \mathbf{V}}\int_{\Sigma_3}\int d^3x \, \mathrm{tr}\left[\gamma^5 G_W \star dQ_W \star G_W \wedge \star dQ_W \star G_W \star \wedge dQ_W\right]$$

Σ_3

$p_4 = \pm\varepsilon \to 0$



Surface Σ_3 surrounds the singularities of $\left[\gamma^5 G_W^{(0)} \star dQ_W^{(0)} \star G_W^{(0)} \wedge \star dQ_W^{(0)} \star G_W^{(0)} \star \wedge dQ_W^{(0)}\right]$

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The Green function entering this expression is the complete one with interactions taken into account

M.Zubkov, R.Abramchuk Physical Review D 107 (9), 094021 (2023)

CSE

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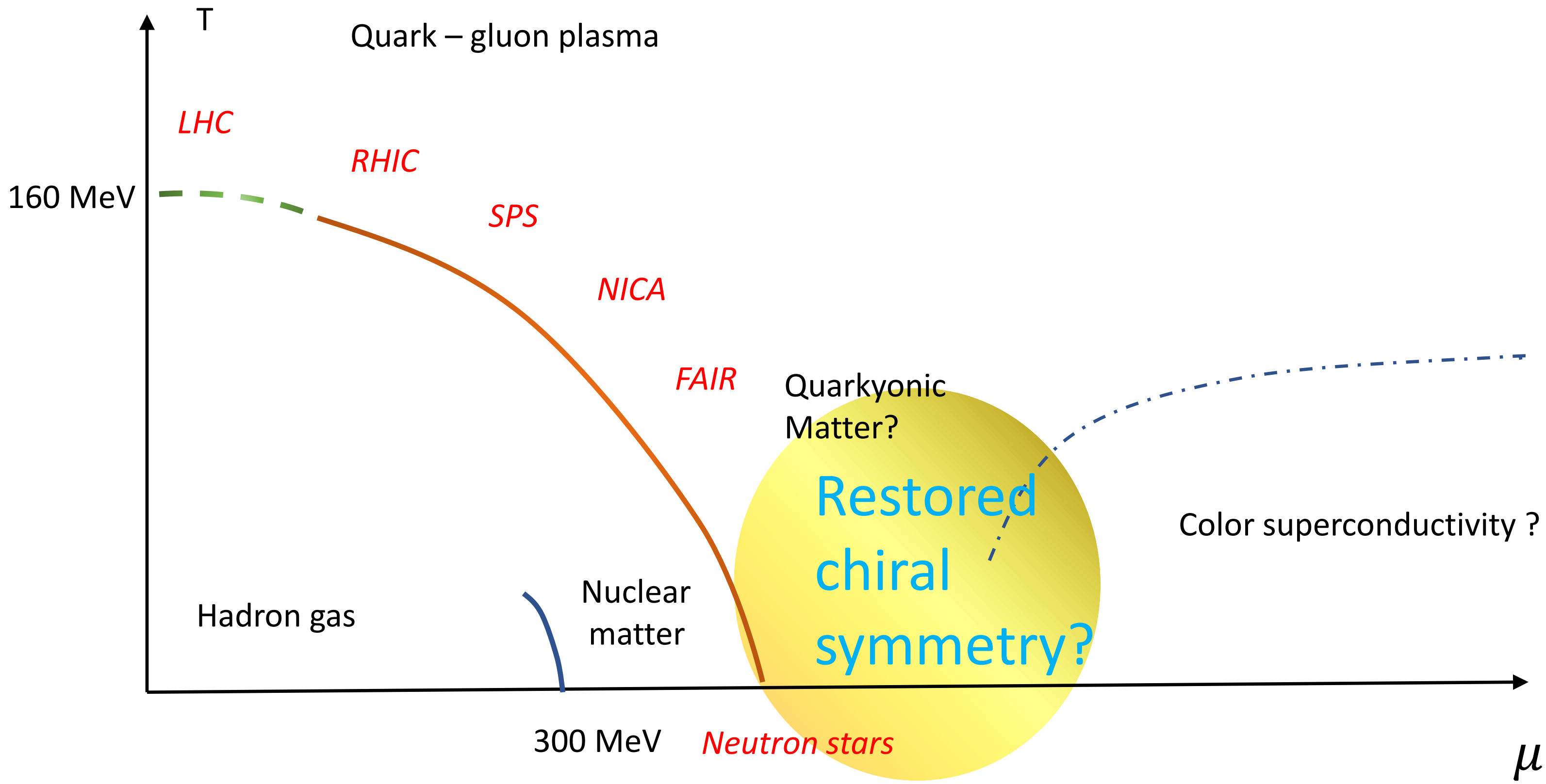
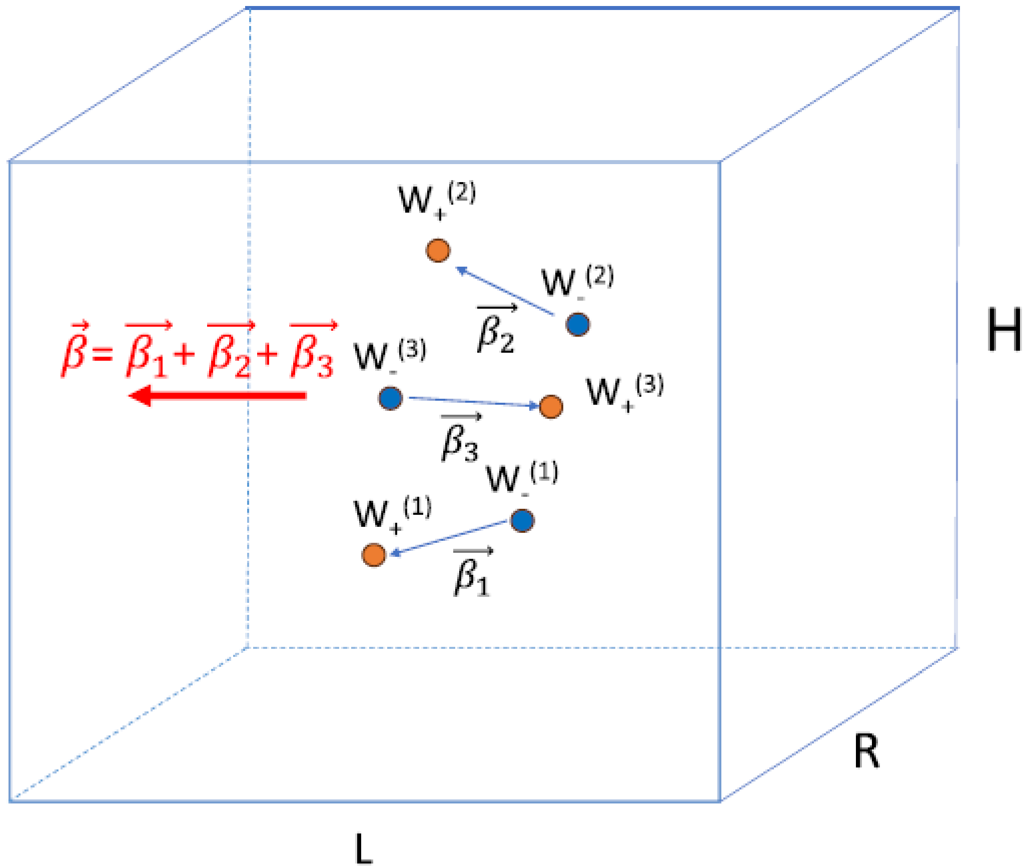


Non – renormalization of CSE by interactions in magnetic Weyl semimetals

CSE



Weyl fermions near Weyl points in momentum space

Non – renormalization of CSE by interactions in magnetic Weyl semimetals

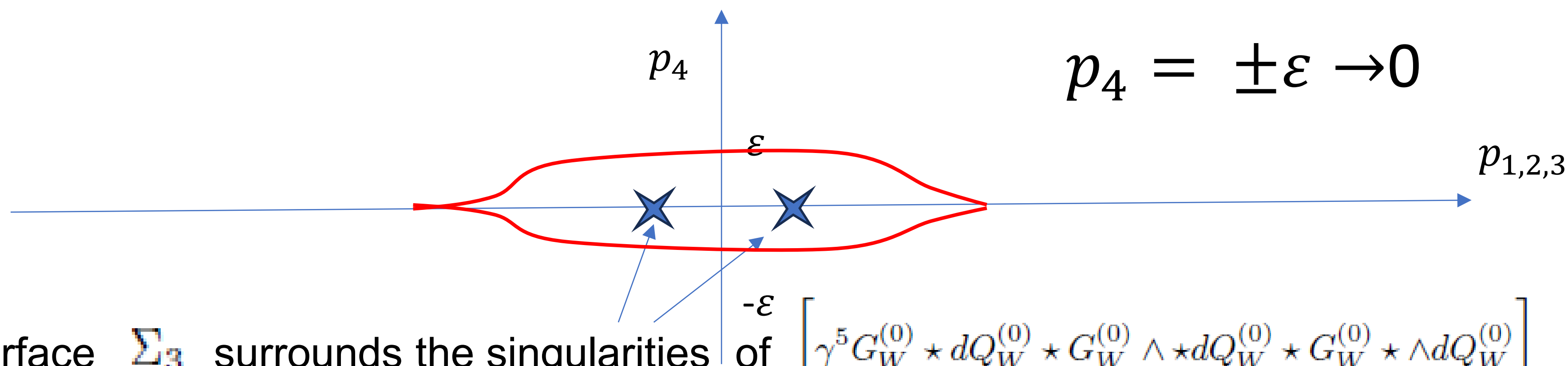
CSE

$$J_k^5(x) = \sigma_{\rm CSE} B_k \delta\mu \qquad \sigma_{\rm CSE} = \frac{\mathcal{N}}{2\pi^2}$$

Chemical potential is counted from the level of Weyl point

$$\mathcal{N} = -\frac{1}{48\pi^2 \mathbf{V}} \int_{\Sigma_3} \int d^3x \, {\rm tr} \left[\gamma^5 G_W \star dQ_W \star G_W \wedge \star dQ_W \star G_W \star \wedge dQ_W\right]$$

Surface Σ_3 surrounds the positions of Weyl points



$p_4 = \pm\varepsilon \to 0$

Surface Σ_3 surrounds the singularities of $\left[\gamma^5 G_W^{(0)} \star dQ_W^{(0)} \star G_W^{(0)} \wedge \star dQ_W^{(0)} \star G_W^{(0)} \star \wedge dQ_W^{(0)}\right]$

γ^5 commutes/anticommutes with Q in small vicinity of the singularities

Non – renormalization of CSE by interactions in magnetic Weyl semimetals

CSE

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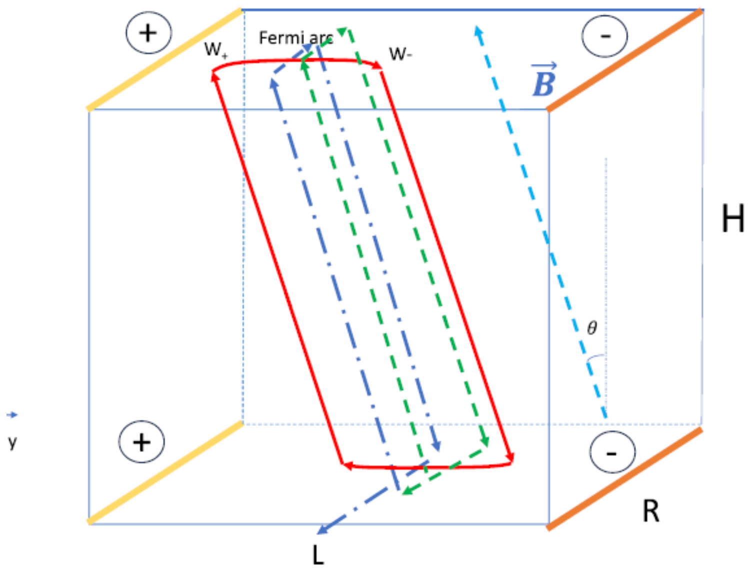
Surface Σ_3 surrounds the positions of Weyl points

The Green function entering this expression is the complete one with interactions taken into account

M A Zubkov 2024 *J. Phys.: Condens. Matter* 36 415501

Proposal for experimental detection of CSE in magnetic Weyl semimetals

CSE



Contribution to QHE conductivity due to the CSE

$$\Sigma_{xy}^{\text{Weyl}} = 2\frac{e\left(\mu-\mu_0\right)\beta}{4\pi^2\hbar^2}\,\frac{1}{B_\perp v_F^{(s)}}L.$$

Chiral Vortical Effect

CVE

as particular case of the CSE

$$S = \int d^4x \bar{\psi}(\gamma^\mu(i\partial_\mu + \mu u_\mu) - M)\psi$$

U is the four – velocity of rotation ➔ effective gauge field

$$u^\mu = \gamma(r)(1, -\Omega y, \Omega x, 0)^T, \quad \gamma(r) = \frac{1}{\sqrt{1-\Omega^2 r^2}}$$

$$A_\mu = -\mu u_\mu$$

Effective magnetic field

$$\mathbf{B} = -(\nabla \times (\mu \mathbf{u}(r)))$$

Weak rotation $\gamma \approx 1$ $\quad \mu_{lab} = \mu\gamma(r) \approx \mu$

$$\mathbf{B} = (0, 0, -2\mu\Omega)$$

Abramchuk, Ruslan, Z. V. Khaidukov, and M. A. Zubkov. "Anatomy of the chiral vortical effect." Physical Review D 98.7 (2018): 076013.

Now we can use the obtained results for the CSE to obtain

the conductivity of the CVE $\quad \bar{J}_5^k = -\frac{\mathcal{N}}{4\pi^2}\epsilon^{ijk0}\mu F_{ij}$ ➔ $j^{5k} = \frac{\mathcal{N}\epsilon^{12k}}{2\pi^2}\mu^2\Omega.$

with

$$\mathcal{N} = -\frac{1}{48\pi^2 \mathbf{V}} \int_{\Sigma_3} \int d^3x \, \mathrm{tr}\left[\gamma^5 G_W \star dQ_W \star G_W \wedge \star dQ_W \star G_W \star \wedge dQ_W\right]$$

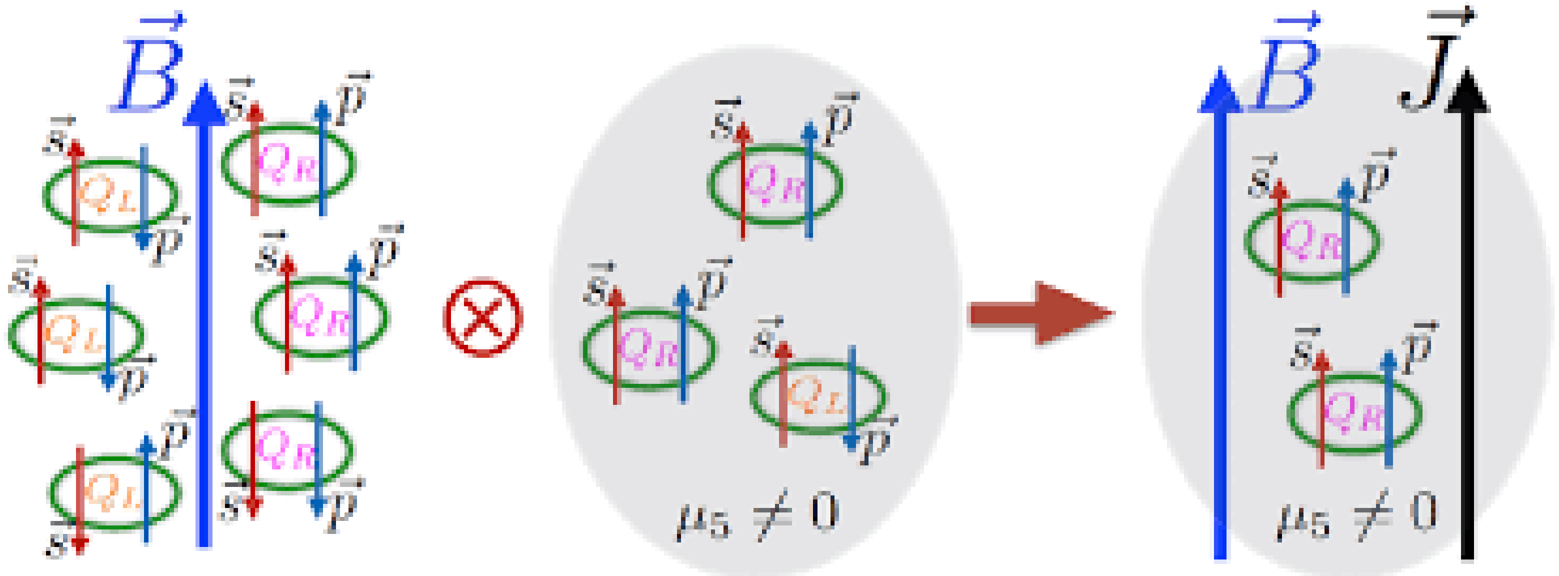
(rotation around the third axis)

Applications to Chiral Magnetic Effect CME

non-homogeneous system, equilibrium, T=0

Average electric current

3 + 1 D:



D.E. Kharzeev, J. Liao, S.A. Voloshin, G. Wang,
Progress in Particle and Nuclear Physics, Volume 88, 2016, Pages 1-28,

Applications to Chiral Magnetic Effect **CME**

non-homogeneous system, equilibrium, T=0

Average electric current

3 + 1 D:

$$\bar{J}^k = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l F_{ij}$$

topological invariant:

$$\mathcal{M}_l = \frac{-iT\epsilon_{ijkl}}{3!V8\pi^2}\int d^D x \int_{\mathcal{M}} d^D p \,\mathrm{Tr}\left[G_W^{(0)} \star \partial_{p_i} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_j} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_k} Q_W^{(0)}\right]$$

external magnetic field: $F_{ij} = \epsilon_{ijk} B_k$

C. Banerjee, M. Lewkowicz, M.A. Zubkov
Physics Letters B, 136457 (2021)

Homogeneous systems: M.A.Zubkov, Physical Review D 93 (10), 105036 (2016)

Chiral magnetic effect ***Equilibrium, T=0*** **CME**

non-homogeneous system

Average electric current

$$\bar{J}^k = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l F_{ij}$$

$$\mathcal{M}_l = \frac{-iT\epsilon_{ijkl}}{3!V8\pi^2}\int d^Dx \int_{\mathcal{M}} d^Dp \mathrm{Tr}\left[G_W^{(0)} \star \partial_{p_i} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_j} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_k} Q_W^{(0)}\right]$$

smooth deformation of the system

⇩

the system without any inhomogeneity

M is not changed!

We know that in homogeneous systems M = 0

Absence of equilibrium chiral magnetic effect, M.A. Zubkov
Physical Review D 93 (10), 105036

⇩

No CME in non – uniform systems at T=0

Applications to Chiral Magnetic Effect

CME

non-homogeneous system, equilibrium, T>0

Average electric current

$$\bar{J}^k = \frac{1}{4\pi^2}\epsilon^{ijk4}\mathcal{M}_4 F_{ij}$$

topological invariant:

$$\mathcal{M}_4 = 2\pi T \sum_{\omega} \mathcal{N}_4(\omega) \qquad \omega = 2\pi T(n+1/2),\ n \in Z,\ 0 \le n < N, \text{ where } N = 1/T$$

$$\mathcal{N}_4(\omega) = \frac{-i\epsilon_{ijk4}}{3!V8\pi^2}\int d^{D-1}x \int_{\mathcal{B}} d^{D-1}p \,\mathrm{Tr}\left[\mathrm{G_W^{(0)}} \star \partial_{p_i}\mathrm{Q_W^{(0)}}(p,x) \star \mathrm{G_W^{(0)}} \star \partial_{p_j}\mathrm{Q_W^{(0)}}(p,x) \star \mathrm{G_W^{(0)}} \star \partial_{p_k}\mathrm{Q_W^{(0)}}\right]$$

Response of N to chiral chemical potential is zero

No CME at T>0

C. Banerjee, M. Lewkowicz, M.A. Zubkov
Physics Letters B, 136457 (2021)

The absence of CME at T>0 **for homogeneous** systems has been reported earlier in
C.G. Beneventano, M. Nieto, E.M. Santangelo J. Phys. A, 53 (46) (2020), Article 465401,

Chiral Magnetic Effect **non-equilibrium systems** CME

Keldysh technique

Green functions (lower sign for fermions)

$$\left\{\hat{G}^R\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv -i\theta(t_1-t_2)\left\langle\left[\Psi_{\alpha_1}(x_1),\Psi^\dagger_{\alpha_2}(x_2)\right]_{+}\right\rangle$$

$$\left\{\hat{G}^A\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv \ i\theta(t_2-t_1)\left\langle\left[\Psi_{\alpha_1}(x_1),\Psi^\dagger_{\alpha_2}(x_2)\right]_{+}\right\rangle$$

$$\left\{\hat{G}^K\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv -i\left\langle\left[\Psi_{\alpha_1}(x_1),\Psi^\dagger_{\alpha_2}(x_2)\right]_{-}\right\rangle,$$

$$\left\{\hat{G}^<\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv -i\left\langle\Psi^\dagger_{\alpha_2}(x_2)\Psi_{\alpha_1}(x_1)\right\rangle$$

Keldysh Green function

$$\hat{G}(t,x|t',x') = -i\begin{pmatrix} \langle T\Phi(t,x)\Phi^+(t',x')\rangle & -\langle\Phi^+(t',x')\Phi(t,x)\rangle \\ \langle\Phi(t,x)\Phi^+(t',x')\rangle & \langle\tilde{T}\Phi(t,x)\Phi^+(t',x')\rangle \end{pmatrix}$$

$$\begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix}$$

$$G^{\mathrm{A}} = G^{--} - G^{+-} = G^{-+} - G^{++}$$

$$G^{\mathrm{R}} = G^{--} - G^{-+} = G^{+-} - G^{++}$$

$$G^< \qquad G^{-+}$$

Keldysh technique and Wigner – Weyl calculus.

Keldysh Green function

$$\hat{G}(t,x|t',x') = -i \begin{pmatrix} \langle T\Phi(t,x)\Phi^+(t',x')\rangle & -\langle \Phi^+(t',x')\Phi(t,x)\rangle \\ \langle \Phi(t,x)\Phi^+(t',x')\rangle & \langle \tilde{T}\Phi(t,x)\Phi^+(t',x')\rangle \end{pmatrix}$$

$$= \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix}$$

$$G^{\mathrm{A}} = G^{--} - G^{+-} = G^{-+} - G^{++}$$

$$G^{\mathrm{R}} = G^{--} - G^{-+} = G^{+-} - G^{++}$$

Wigner transformation

$$G^{<} = G^{-+}$$

$$\hat{G}(X_1, X_2) = \langle X_1|\hat{\mathbf{G}}|X_2\rangle \qquad A(X_1, X_2) = \langle X_1|\hat{A}|X_2\rangle$$

$$A_W(X|P) = \int d^{D+1}Y\, e^{\mathrm{i}Y^\mu P_\mu} A(X+Y/2, X-Y/2)$$

Moyal product

$$(A\star B)(X|P) = A(X|P)\, e^{-\mathrm{i}(\overleftarrow{\partial}_{X^\mu}\overrightarrow{\partial}_{P_\mu} - \overleftarrow{\partial}_{P_\mu}\overrightarrow{\partial}_{X^\mu})/2} B(X|P)$$

Lesser representation

$$\hat{\mathbf{G}}^{(<)} = U\hat{\mathbf{G}}V,$$

CME

U= $\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ V = $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

$$\hat{\mathbf{G}}^{(<)} = \begin{pmatrix} G^{\mathrm{R}} & 2G^{<} \\ 0 & G^{\mathrm{A}} \end{pmatrix}$$

$$G^{<} = G^{-+}$$

$$G^{\mathrm{A}} = G^{--} - G^{+-} = G^{-+} - G^{++}$$

$$G^{\mathrm{R}} = G^{--} - G^{-+} = G^{+-} - G^{++}$$

The inverse Q of Green function

$$\hat{\mathbf{Q}}\hat{\mathbf{G}} = 1$$

After Wigner transformation

$$\hat{Q} * \hat{G} = 1$$

Response of electric current to external field strength

$$J^i = -\frac{1}{4}\int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \,\mathrm{tr}\left(\hat{G}\star\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\partial_{\pi_i}\hat{Q}\right)^{<}\mathcal{F}^{\mu\nu}$$
$$-\frac{1}{4}\int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \,\mathrm{tr}\left(\partial_{\pi_i}\hat{Q}\hat{G}\star\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\right)^{<}\mathcal{F}^{\mu\nu}.$$

Electric conductivity tensor for non – homogeneous systems

$$J^i = \sigma^{ij}\mathcal{F}_{0j}$$

$$\sigma^{ij} = \frac{1}{4}\int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \,\mathrm{tr}\left(\partial_{\pi_i}\hat{Q}\left[\hat{G}\star\partial_{\pi_{[0}}\hat{Q}\star\partial_{\pi_{j]}}\hat{G}\right]\right)^{<} + \mathrm{c.c.}$$

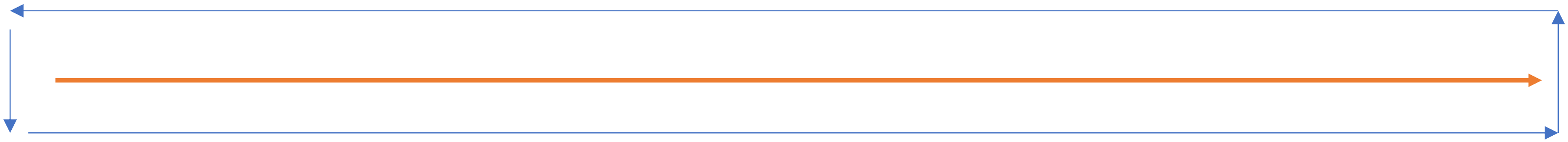
C Banerjee, IV Fialkovsky, M Lewkowicz, CX Zhang, MA Zubkov
Journal of Computational Electronics 20, 2255-2283 (2021)

2D Hall conductivity
“Topological part”

$$\bar{\sigma}_H = -\frac{\mathcal{N}_f}{2\pi} + \bar{\sigma}_{H,f'}$$

$$\mathcal{N}_f = -\frac{1}{48\pi^2\,\mathcal{V}}\epsilon^{\mu\nu\rho}\oint d\pi^0 \int d^2\pi d^2x \;\mathrm{tr}\left(\partial_{\pi^\mu} Q \star \partial_{\pi^\nu} G \star \partial_{\pi^\rho} Q \star G\right) f(\pi^0) + \mathrm{c.c.}$$

C Banerjee, IV Fialkovsky, M Lewkowicz, CX Zhang, MA Zubkov, arXiv:2009.10704



A similar expression has been obtained independently in F.R. Lux, F. Freimuth, S. Bl¨ugel, Y. Mokrousov, Physical Review Letters 124 (9), 096602 (2020)

contour in complex plane of $\boldsymbol{\pi^0}$
in the case of thermal equilibrium at T->0

$$\mathcal{N}_f = -\frac{1}{24\pi^2\,\beta\,\mathcal{V}}\epsilon^{\mu\nu\rho}\int d^3X \int d^3\Pi \;\mathrm{tr}\left(\partial_{\Pi^\mu}\hat{Q}^{\mathrm{M}} \star \hat{G}^{\mathrm{M}} \star \partial_{\Pi^\nu}\hat{Q}^{\mathrm{M}} \star \hat{G}^{\mathrm{M}} \star \partial_{\Pi^\rho}\hat{Q}^{\mathrm{M}} \star \hat{G}^{\mathrm{M}}\right)$$

Matsubara Green function

$\boldsymbol{G^M}$ ***(we replace inside*** $\boldsymbol{G^R \quad \pi^0 \to i\,\omega)}$

2D Hall conductivity

$$\bar{\sigma}_H = -\frac{\mathcal{N}_f}{2\pi} + \bar{\sigma}_{H,f'}$$

"non - topological part"

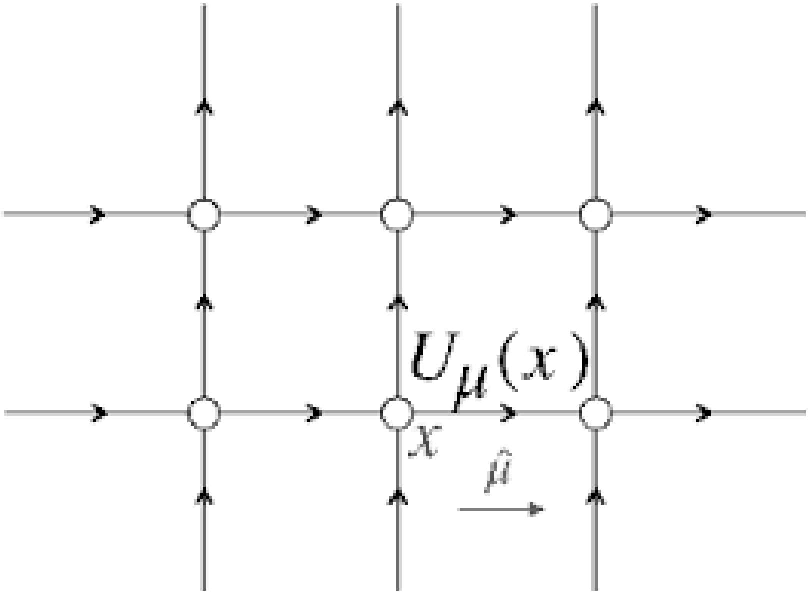
$$\bar{\sigma}_{H,f'} = +\frac{1}{8\,\mathcal{V}}\epsilon^{ij}\int \frac{d^3\pi d^2x}{(2\pi)^3}\,\mathrm{tr}\left((\partial_{\pi^i}Q^{\mathrm{R}}\star G^{\mathrm{R}} + \partial_{\pi^i}Q^{\mathrm{A}}\star G^{\mathrm{A}})\star\partial_{\pi^j}Q^{\mathrm{R}}\star(G^{\mathrm{A}} - G^{\mathrm{R}})\right)\partial_{\pi^0}f(\pi^0) + c.c.$$

ordinary symmetric conductivity

$$\bar{\sigma}_{\parallel}^{ij} = \frac{1}{8\,\mathcal{V}}\int \frac{d^3\pi d^2x}{(2\pi)^3}\,\mathrm{tr}\left((-\partial_{\pi^i}Q^{\mathrm{R}}\star G^{\mathrm{R}} + \partial_{\pi^i}Q^{\mathrm{A}}\star G^{\mathrm{A}})\star\partial_{\pi^j}Q^{\mathrm{R}}\star(G^{\mathrm{A}} - G^{\mathrm{R}})\right)\partial_{\pi^0}f(\pi^0) + (i\leftrightarrow j) + \mathrm{c.c.}$$

Lattice model with Wilson fermions

Out of equilibrium



Thermal equilibrium (in Euclidean space - time)

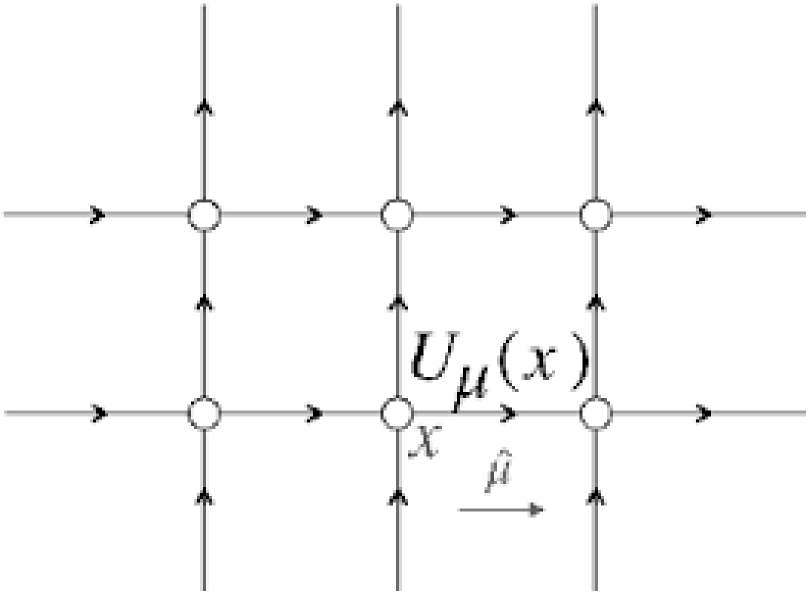
$$Q_W^M(\pi) = \sum_{\mu=1}^{3} \gamma^\mu g_\mu(\pi) - im(\pi) + \gamma^4 g_4(\pi_4)$$

$$g_i = \sin(\pi_i)$$

$$m(\pi) = m^{(0)} + \sum_{i=1}^{4} (1 - \cos(\pi_i))$$

Lattice model with Wilson fermions

Out of equilibrium



Real time dynamics (in Minkowski space - time)

$$Q^M_W(\pi)|_{\pi_4=-i\pi_0} = \sum_{\mu=1}^{3} \gamma^\mu g_\mu(\pi) \qquad ($$

$$-i\left(\sum_{i=1}^{3}(1-\cos(\pi_i)) + (1-\mathrm{ch}(\pi_0))\right) - i\gamma^4 \mathrm{sh}(\pi_0)$$

CME

Lattice model with Wilson fermions

Out of equilibrium

Keldysh Green function

$$\hat{Q} = \begin{pmatrix} Q_{--} & Q_{-+} \\ Q_{+-} & Q_{++} \end{pmatrix}$$

$$Q_{++} = -\mathcal{Q}(\pi_0,\vec{\pi}) + \mathrm{i}\epsilon\partial_{\pi_0}\mathcal{Q}(\pi_0,\vec{\pi})\frac{1-\rho(\pi_0)}{1+\rho(\pi_0)}$$

$$Q_{--} = \mathcal{Q}(\pi_0,\vec{\pi}) + +\mathrm{i}\epsilon\partial_{\pi_0}\mathcal{Q}(\pi_0,\vec{\pi})\frac{1-\rho(\pi_0)}{1+\rho(\pi_0)}$$

$$Q_{+-} = -2\mathrm{i}\epsilon\partial_{\pi_0}\mathcal{Q}(\pi_0,\vec{\pi})\frac{1}{1+\rho(\pi_0)},$$

$$Q_{-+} = 2\mathrm{i}\epsilon\partial_{\pi_0}\mathcal{Q}(\pi_0,\vec{\pi})\frac{\rho(\pi_0)}{1+\rho(\pi_0)}.$$

$$\pi = P - A(X)$$

initial one – particle distribution

$$f(\pi_0) = \rho(\pi_0)(1+\rho(\pi_0))^{-1}$$

CME

$$\hat{Q} = \begin{pmatrix} Q_{--} & Q_{-+} \\ Q_{+-} & Q_{++} \end{pmatrix}$$

$$Q_{++} = -\Big(\sum_{\mu=1}^{3} \gamma^\mu g_\mu(\pi) - im(\vec{\pi}, -i\pi_0 - i\mu_5(t)\gamma^5)$$

$$+\gamma^4 g_4(-i\pi_0 - i\mu_5(t)\gamma^5) - \gamma^4 \epsilon\, e^{-\pi_0\gamma^4} \frac{1-\rho(\pi_0)}{1+\rho(\pi_0)}\Big),$$

$$Q_{--} = \sum_{\mu=1}^{3} \gamma^\mu g_\mu(\pi) - im(\vec{\pi}, -i\pi_0 - i\mu_5(t)\gamma^5)$$

$$+\gamma^4 g_4(-i\pi_0 - i\mu_5(t)\gamma^5) + \gamma^4 \epsilon\, e^{-\pi_0\gamma^4} \frac{1-\rho(\pi_0)}{1+\rho(\pi_0)},$$

$$Q_{+-} = -2\gamma^4 \epsilon\, e^{-\pi_0\gamma^4} \frac{1}{1+\rho(\pi_0)},$$

$$Q_{-+} = 2\gamma^4 \epsilon\, e^{-\pi_0\gamma^4} \frac{\rho(\pi_0)}{1+\rho(\pi_0)}. \qquad (32)$$

time depending chiral chemical potential

$$\delta\mu_5(t) = \delta\mu_5^{(0)} \cos\omega_0 t$$

Response of electric current both to magnetic field and to chiral chemical potential

CME

$$J^i = \Sigma_{CME} B^i$$

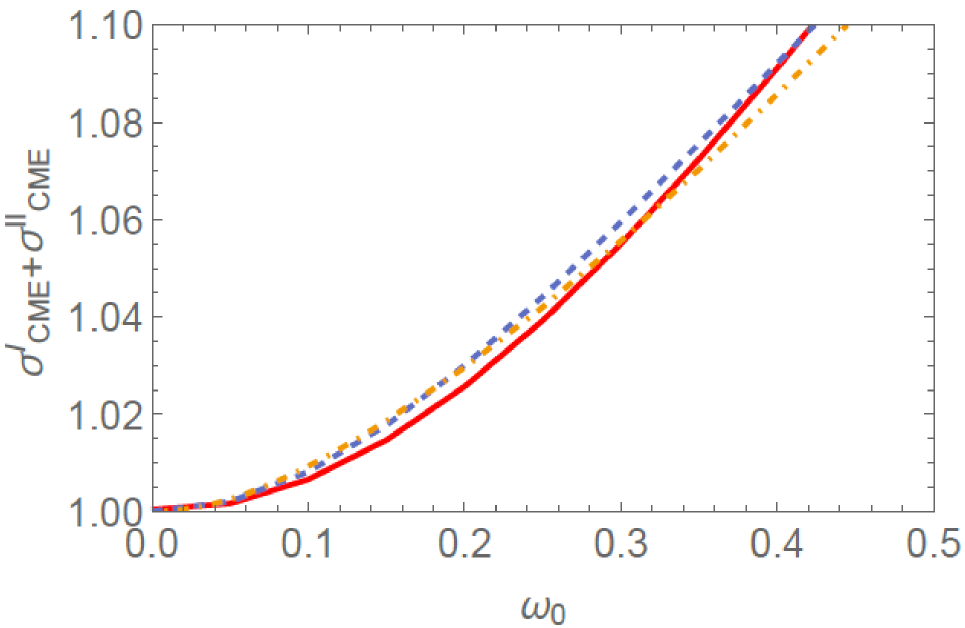
response to chiral chemical potential

$$\delta\mu_5(t) = \delta\mu_5^{(0)} \cos\omega_0 t$$

$$\Delta\Sigma_{CME} = \frac{1}{4\pi^2}\sigma_{CME}(\omega_0)\delta\mu_5^{(0)} e^{\mathrm{i}\omega_0 t} + (c.c.)$$

two parts of conductivity

$$\sigma_{CME}(\omega_0) = \sigma_{CME}^{(I)}(\omega_0) + \sigma_{CME}^{(II)}(\omega_0)$$



$T = \frac{1}{10a}$ (solid line), $\frac{1}{20a}$ (dashed line), $\frac{1}{50a}$ (dashed - dotted line)

Response of electric current both to magnetic field and to chiral chemical potential

CME

$$J^i = \Sigma_{CME} B^i$$

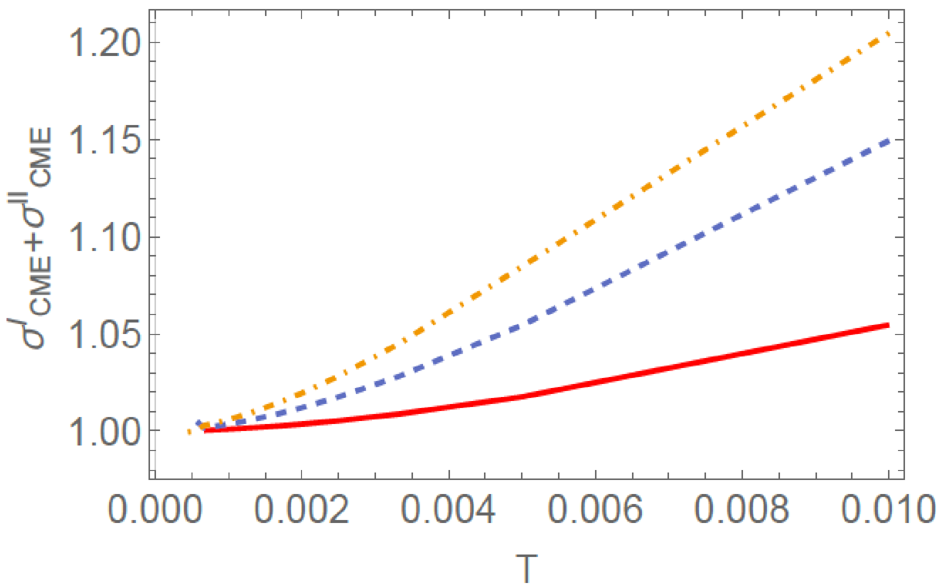
response to chiral chemical potential

$$\delta\mu_5(t) = \delta\mu_5^{(0)} \cos\omega_0 t$$

$$\Delta\Sigma_{CME} = \frac{1}{4\pi^2}\sigma_{CME}(\omega_0)\delta\mu_5^{(0)} e^{i\omega_0 t} + (c.c.)$$

two parts of conductivity

$$\sigma_{CME}(\omega_0) = \sigma_{CME}^{(I)}(\omega_0) + \sigma_{CME}^{(II)}(\omega_0)$$



C. Banerjee, M. Lewkowicz, M.A. Zubkov
Physical Review D 106 (7), 074508 (2022)

$x = \omega_0/T = 30$ (solid line), $x = 60$ (dashed line), $x = 80$ (dashed dotted line)

Out of equilibrium the CME is back!!!

When chiral chemical potential is time dependent, the CME conductivity depends on frequency w. In the continuum limit the conventional value of CME conductivity is reproduced for any ratio w/T.

The absence of interaction corrections to Quantum Hall Effect

Electric current orthogonal to electric field in the presence of magnetic field

$$S = \int d\tau \sum_{\mathbf{x},\mathbf{x}'} [\bar{\psi}_{\mathbf{x}'}(i(i\partial_\tau - A_3(i\tau,\mathbf{x}))\delta_{\mathbf{x},\mathbf{x}'} - i\mathcal{D}_{\mathbf{x},\mathbf{x}'})\psi_{\mathbf{x}}$$

$$+ \alpha\bar{\psi}(\tau,\mathbf{x})\psi(\tau,\mathbf{x})\theta(y)V(\mathbf{x}-\mathbf{x}')\theta(y')\bar{\psi}(\tau,\mathbf{x}')\psi(\tau,\mathbf{x}')]$$

as an example:

$$\mathcal{D}_{\mathbf{x},\mathbf{x}'} = -\frac{i}{2}\sum_{i=1,2}[(1+\sigma^i)\delta_{x+e_i,x'}e^{iA_{x+e_i,x}}$$

$$+ (1-\sigma^i)\delta_{x-e_i,x'}e^{iA_{x-e_i,x}}]\sigma_3 + i(m+2)\delta_{\mathbf{x},\mathbf{x}'}\sigma_3$$

without interactions:

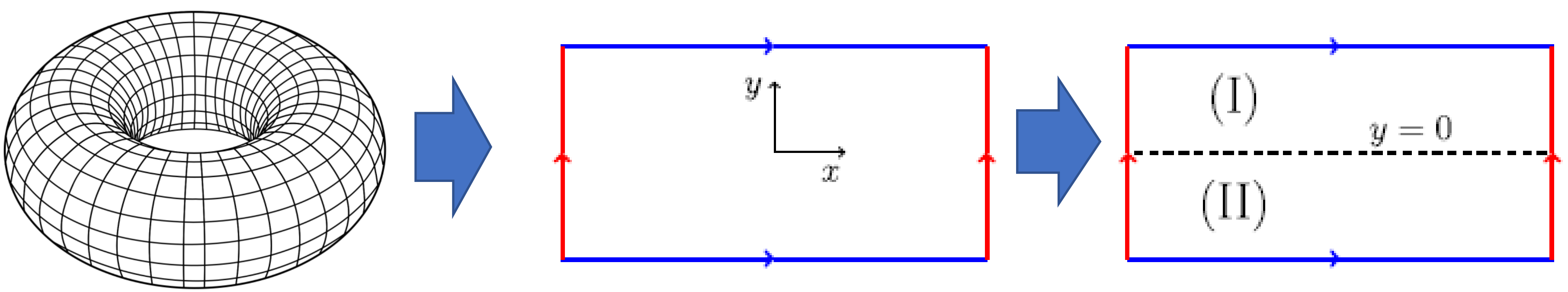
$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$

$$\mathcal{N} = \frac{T}{S3!4\pi^2}\epsilon_{ijk}\int d^3x \int d^3p \,\mathrm{Tr}\, G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}.$$

$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$

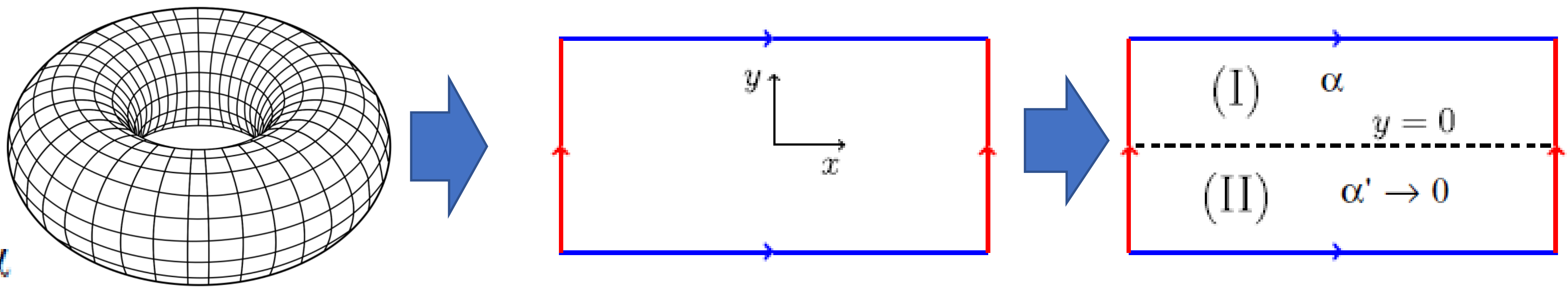
$$\mathcal{N} = \frac{T}{S3!4\pi^2}\epsilon_{ijk}\int d^3x \int d^3p \,\mathrm{Tr}\, G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}.$$

Gedankenexperiment:
we consider the system on the torus
and divide it into the two pieces



we consider the system on the torus and divide it into the two pieces

$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$



α is zero in the part II, E(I) = - E(II)

$$I_{tot} = (I_1 + I_2)/2 = (\bar{\sigma}_1 E + \bar{\sigma}_2(-E))/2 + I_{tot}\Big|_{E=0}$$

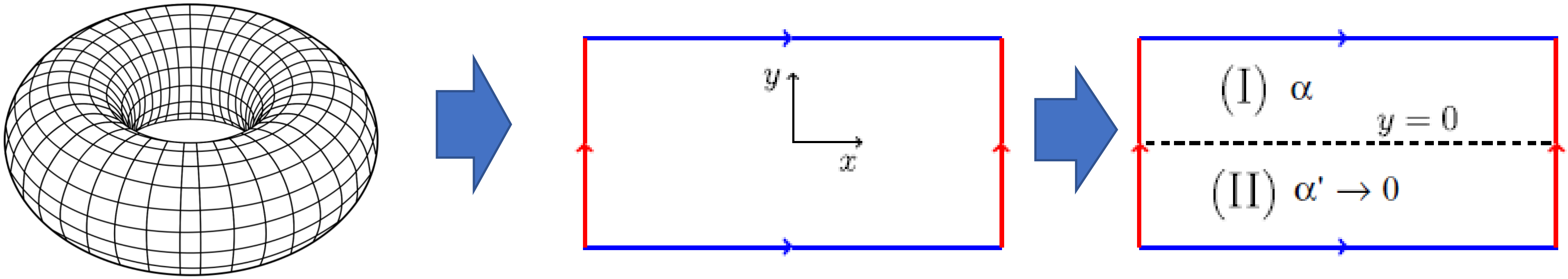
$$\bar{\sigma}_1(0) = \bar{\sigma}_2(0),$$

We prove that the total current remains zero with the interaction corrections

$$\bar{\sigma}_1(g) = \bar{\sigma}_2(0)$$

$$\bar{\sigma}_1(0) = \bar{\sigma}_1(g)$$

no interaction corrections



is zero in the part II, E(I) = - E(II)

$$I(g) = \int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\, G_{g,W}(R,p) \star \frac{\partial}{\partial p_x} Q_W(R,p)$$

$$G_{g,W} = G_W + G_W \star \Sigma_W \star G_W + \ldots$$

$$I(g) = \sum_{n=0}^{\infty} I^{(n)}$$

$$I^{(n)} = \int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\, (G_W \star \Sigma_W \star)^n G_W \star \frac{\partial Q_W}{\partial p_x}$$

an example: 1-loop

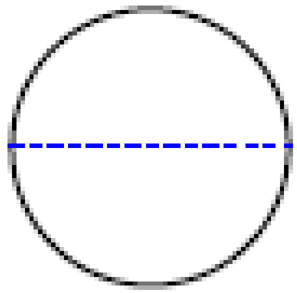
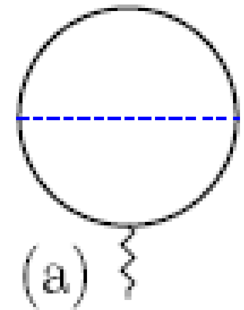
$$\mathcal{I}_1 = -\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\Big[\int \frac{d^3q}{(2\pi)^3} G_W(R, p-q)\mathcal{D}(q)\Big] \star \frac{\partial}{\partial p_x} G_W(R,p)$$

$$-\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr} \frac{\partial}{\partial p_x}\Big[G_W(R, p-q) \star G_W(R,p)\Big]\mathcal{D}(q) = 0$$

$$I(g) = \sum_{n=0}^{\infty} I^{(n)}$$

$$I^{(n)} = \int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\left(G_W \star \Sigma_W \star\right)^n G_W \star \frac{\partial Q_W}{\partial p_x}$$

an example: 1-loop diagram

$\frac{\partial}{\partial p_x}$  = 2 

(a)

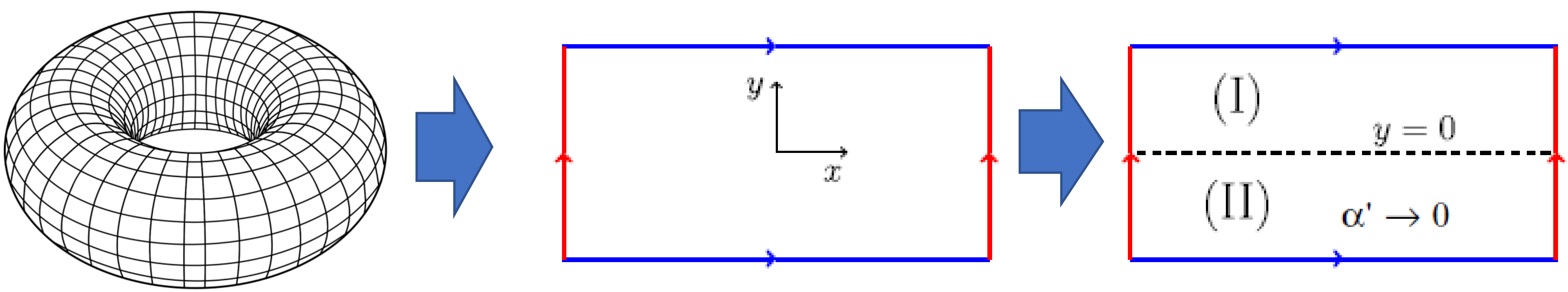
$$\mathcal{I}_1 = -\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\Big[\int \frac{d^3q}{(2\pi)^3} G_W(R, p-q)\mathcal{D}(q)\Big] \star \frac{\partial}{\partial p_x} G_W(R, p)$$

$$-\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\frac{\partial}{\partial p_x}\Big[G_W(R, p-q) \star G_W(R, p)\Big]\mathcal{D}(q) = 0.$$

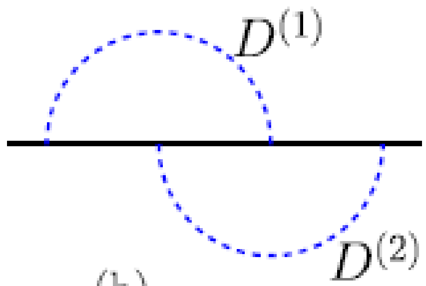
$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$

$$\mathcal{N} = \frac{T}{S3!4\pi^2}\epsilon_{ijk}\int d^3x\int d^3p\,\mathrm{Tr}\,G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}.$$

In the presence of interactions the sum of the currents in the two pieces is zero ➔ the electric conductivity receives no corrections in the part I



Another example of diagram technique



(b)

$$\int\int [G_1(R,p) \circ_1 \star G_2(R,p-k_1) \circ_2 \star G_3(R,p-k_1-k_2) \star_1 \circ G_4(R,p-k_2) \star_2 \circ G_5(R,p)]$$
$$D_W^{(1)}(R,k_1) D_W^{(2)}(R,k_2) dk_1 dk_2.$$

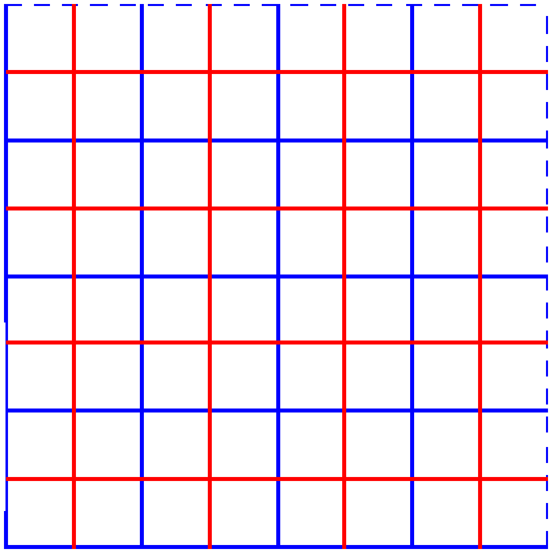
$\circ_j = e^{-i\overleftarrow{\partial}_p \partial_R^{(j)}/2}$ and ${}_j\circ = e^{i\partial_R^{(j)}\overrightarrow{\partial}_p/2}$. $\partial_R^{(j)}$ acts on $D^{(j)}$ only.

Precise Wigner – Weyl calculus. Finite rectangular lattice

$$\mathcal{O} = \{(m_1, ..., m_D)|m_i \in \{0, 1, 2, ..., N-1\}\}$$

$$\mathcal{O}' = \{(m_1, ..., m_D)|m_i \in \{0, 1/2, 1, ..., N-1/2\}\}$$

$$\mathcal{M} = \{(m_1 \frac{2\pi}{N}, ..., m_D \frac{2\pi}{N})|m_i \in \{0, 1, 2, ..., N-1\}\}$$



refined lattice $\mathcal{O}'$

$$\mathcal{M}' = \{(2\pi m_1/N, ..., 2\pi m_D/N)|m_i \in \{0, 1/2, 1, ..., N-1/2\}\}$$

Weyl symbol of operator

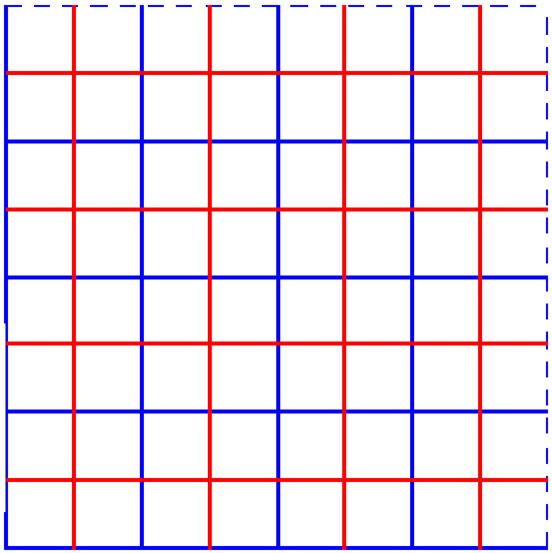
$$A_W(p,q) = \sum_{n_i=0,1; v\in\mathcal{O}'} e^{2ipv} \langle q - v + n/2|\hat{A}|q + v + n/2\rangle$$
$$\Pi_i \frac{1 + e^{2iv_i\pi/(N)}}{2} \frac{1 + e^{2\pi i(q_i - v_i + n_i/2)}}{2}.$$

Precise Wigner – Weyl calculus. Finite rectangular lattice

$$\mathcal{O} = \{(m_1, ..., m_D) | m_i \in \{0, 1, 2, ..., N-1\}\}$$

$$\mathcal{O}' = \{(m_1, ..., m_D) | m_i \in \{0, 1/2, 1, ..., N-1/2\}\}$$

$$\mathcal{M} = \{(m_1 \frac{2\pi}{N}, ..., m_D \frac{2\pi}{N}) | m_i \in \{0, 1, 2, ..., N-1\}\}$$



refined lattice $\mathcal{O}'$

$$\mathcal{M}' = \{(2\pi m_1/N, ..., 2\pi m_D/N) | m_i \in \{0, 1/2, 1, ..., N-1/2\}\}$$

Weyl symbol of operator for continuous arguments

$$A_W(p,q) = \sum_{p_1 \in \mathcal{M}'; q_1 \in \mathcal{O}'; p_2 \in \mathcal{M}'; q_2 \in \mathcal{O}'} \frac{1}{(4N^2)^D} e^{2i((p_2-p)(q_1-q)+(q_2-q)(p-p_1))} A_W(p_1, q_1)$$

Properties of Weyl symbol

$$\mathrm{Tr}\,\hat{A} = \frac{1}{(4N)^D} \sum_{p\in\mathcal{M}',q\in\mathcal{O}'} A_W(p,q)$$

$$\mathrm{Tr}\hat{A}\hat{B} = \frac{1}{(4N)^D} \sum_{p\in\mathcal{M}',q\in\mathcal{O}'} A_W(p,q)B_W(p,q)$$

$$(\hat{A}\hat{B})_W(p,q)\Big|_{p\in\mathcal{M}',q\in\mathcal{O}'} = A_W(p,q)e^{\frac{i}{2}(\overleftarrow{\partial_q}\overrightarrow{\partial_p}-\overleftarrow{\partial_p}\overrightarrow{\partial_q})}B_W(p,q) = A_W(p,q)\star B_W(p,q)$$

$$1_W(p,q)\Big|_{p\in\mathcal{M}',q\in\mathcal{O}'} = 1$$

translation to one lattice spacing

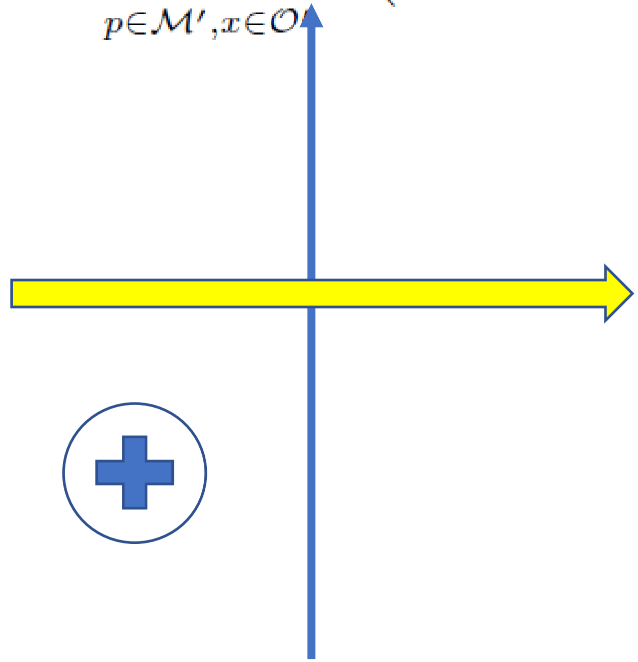
$$T_j(p,q) = e^{ip_j}\Big(\frac{1+e^{2i\pi/N}}{2} + e^{iNp_j}\frac{1-e^{2i\pi/N}}{2}\Big)$$

Applications: QHE

$$\bar{\sigma}^{ij} = \frac{\mathcal{N}}{2\pi}\epsilon^{ij}$$

$$\mathcal{N} = \frac{1}{3!}\epsilon^{\mu\nu\rho}\frac{1}{(2N)^{2D}}\int d\Pi^3 \sum_{p\in\mathcal{M}',x\in\mathcal{O}} \mathrm{tr}\left(\partial_{\Pi^\mu}\hat{Q}^{\mathrm{M}}_W \star \hat{G}^{\mathrm{M}}_W \star \partial_{\Pi^\nu}\hat{Q}^{\mathrm{M}}_W \star \hat{G}^{\mathrm{M}}_W \star \partial_{\Pi^\rho}\hat{Q}^{\mathrm{M}}_W \star \hat{G}^{\mathrm{M}}_W\right)$$

electric current j



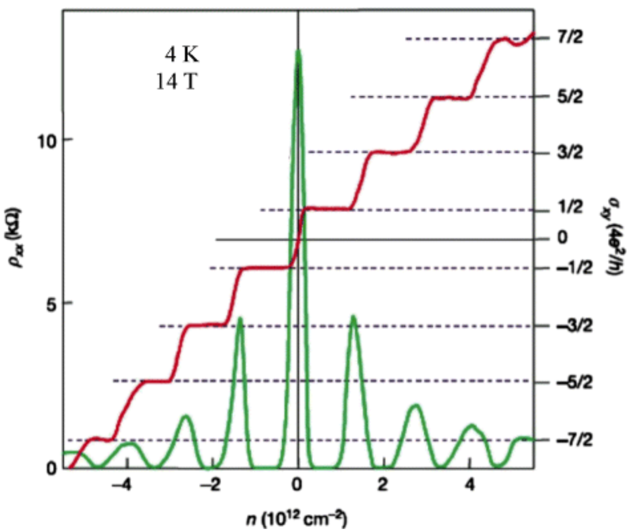
electric field E

M.A. Zubkov (2023)
Journal of Physics A: Mathematical and Theoretical 56 (39), 395201

Quantum Hall Effect

QHE

constant magnetic field, no interactions, no disorder
k is Bloch vector,
|u(k)> is the eigenvector of
Hamiltonian



$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\int d^2k[\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{k})]$$

$$\boldsymbol{A}(\boldsymbol{k}) = -\mathrm{i}\,\langle u(\boldsymbol{k})|\boldsymbol{\nabla}|u(\boldsymbol{k})\rangle\,.$$

TKNN invariant

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs
Phys. Rev. Lett. 49, 405 (1982)

Intrinsic Anomalous Quantum Hall Effect

QHE

homogeneous system

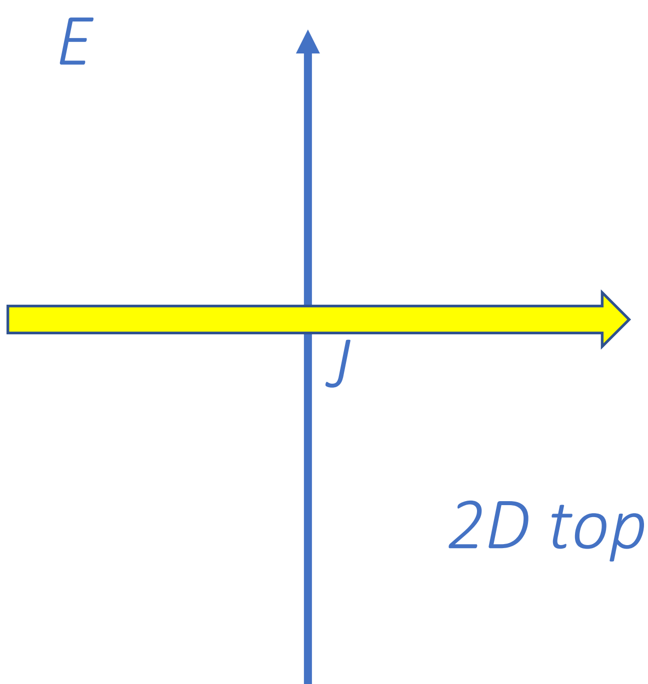
no magnetic field

no interactions

no disorder

T. Matsuyama, Quantization of Conductivity Induced by Topological Structure of Energy Momentum Space in Generalized QED in Three-dimensions, Prog. Theor. Phys 77, 711 (1987)

$$\mathcal{N} = \frac{\epsilon_{ijk}}{3!\,4\pi^2} \int d^3p \, \mathrm{Tr}\left[G(p)\frac{\partial G^{-1}(p)}{\partial p_i}\frac{\partial G(p)}{\partial p_j}\frac{\partial G^{-1}(p)}{\partial p_k}\right]$$



$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

2D topological insulator

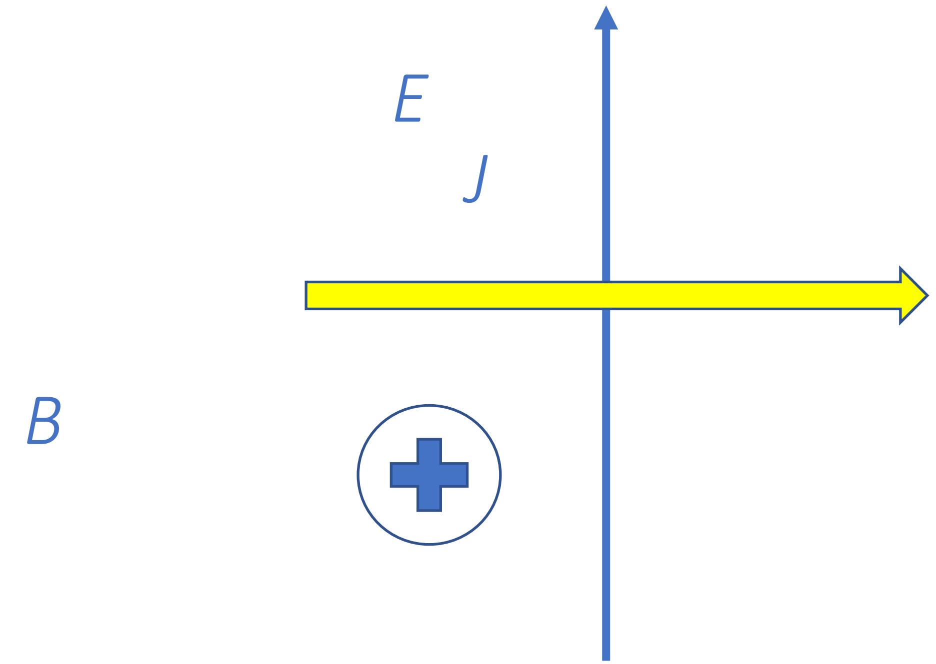
Applications to Quantum Hall Effect

QHE

Equilibrium, T=0

non-homogeneous system

Average electric current



2+1 D:

$$\langle j^k \rangle = -\frac{1}{2\pi} \mathcal{N} \epsilon^{3kj} E_j ;$$

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\,3!\,4\pi^2} \int d^3p d^3x \, \mathrm{Tr} \left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k} \right]$$

M.A. Zubkov [*,1], Xi Wu

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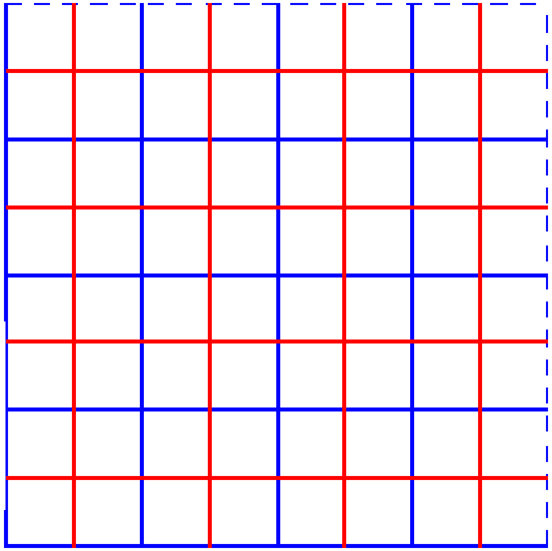
Precise Wigner – Weyl calculus. INFINITE rectangular lattice $N \to$ infinity

$$\mathcal{O} = \{(m_1, ..., m_D) | m_i \in \{0, 1, 2, ..., N-1\}\}$$

$$\mathcal{O}' = \{(m_1, ..., m_D) | m_i \in \{0, 1/2, 1, ..., N-1/2\}\}$$

$$\mathcal{M} = \{(m_1 \frac{2\pi}{N}, ..., m_D \frac{2\pi}{N}) | m_i \in \{0, 1, 2, ..., N-1\}\}$$



refined lattice $\mathcal{O}'$

$$\mathcal{M}' = \{(2\pi m_1/N, ..., 2\pi m_D/N) | m_i \in \{0, 1/2, 1, ..., N-1/2\}\}$$

Weyl symbol of operator (momentum space becomes continuous)

$$A_W(p,q) = \int_{\mathcal{M}} d^D p_- \, \langle\langle p - p_- | \hat{A} | p + p_- \rangle\rangle \, e^{2ip_- q} \Pi_i (1 + e^{ip^i_-})$$

$$\langle\langle p_1 | p_2 \rangle\rangle = \delta(p_1 - p_2).$$

Properties of Weyl symbol N → infinity

$$\mathrm{Tr}\,\hat{A} = \frac{1}{(4N)^D} \sum_{p\in\mathcal{M}',q\in\mathcal{O}'} A_W(p,q)$$

$$\mathrm{Tr}\hat{A}\hat{B} = \frac{1}{(4N)^D} \sum_{p\in\mathcal{M}',q\in\mathcal{O}'} A_W(p,q)B_W(p,q)$$

$$(\hat{A}\hat{B})_W(p,q)\Big|_{p\in\mathcal{M}',q\in\mathcal{O}'} = A_W(p,q)e^{\frac{i}{2}(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q)}B_W(p,q) = A_W(p,q)\star B_W(p,q)$$

$$1_W(p,q)\Big|_{p\in\mathcal{M}',q\in\mathcal{O}'} = 1$$

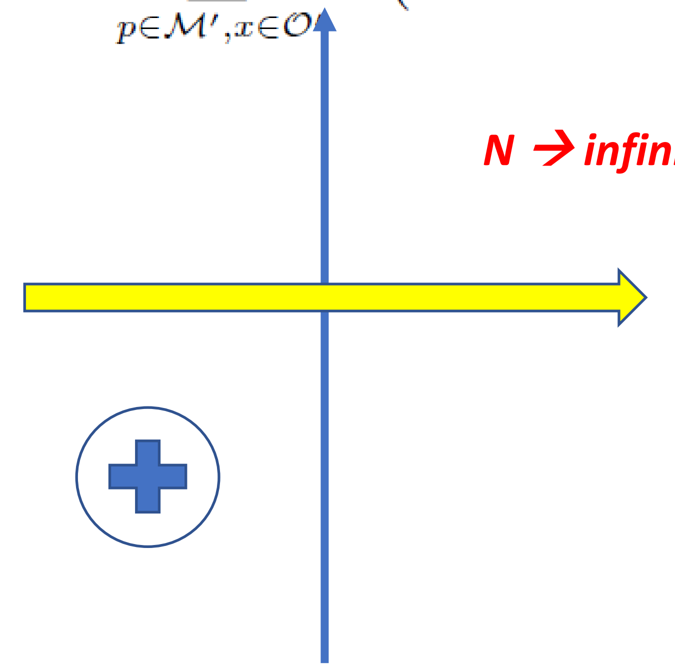
translation to one lattice spacing

$$T_j(p,q) = e^{ip_j}\Big(\frac{1+e^{2i\pi/N}}{2} + e^{iNp_j}\frac{1-e^{2i\pi/N}}{2}\Big) \qquad T_j(p,q) \to e^{ip_j}$$

Applications: QHE

$$\bar{\sigma}^{ij} = \frac{\mathcal{N}}{2\pi}\epsilon^{ij}$$

$$\mathcal{N} = \frac{1}{3!}\epsilon^{\mu\nu\rho}\frac{1}{(2N)^{2D}}\int d\Pi^3 \sum_{p\in\mathcal{M}',x\in\mathcal{O}} \mathrm{tr}\left(\partial_{\Pi^\mu}\hat{Q}^{\mathrm{M}}_W \star \hat{G}^{\mathrm{M}}_W \star \partial_{\Pi^\nu}\hat{Q}^{\mathrm{M}}_W \star \hat{G}^{\mathrm{M}}_W \star \partial_{\Pi^\rho}\hat{Q}^{\mathrm{M}}_W \star \hat{G}^{\mathrm{M}}_W\right)$$



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Nuclear Physics B 954, 114999

Precise Wigner – Weyl calculus. INFINITE HONEYCOMB lattice

N → infinity

coordinate space

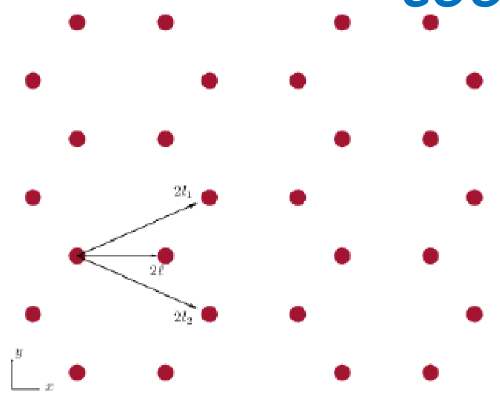


FIG. 2. An illustration of the physical lattice $\mathscr{O}$.

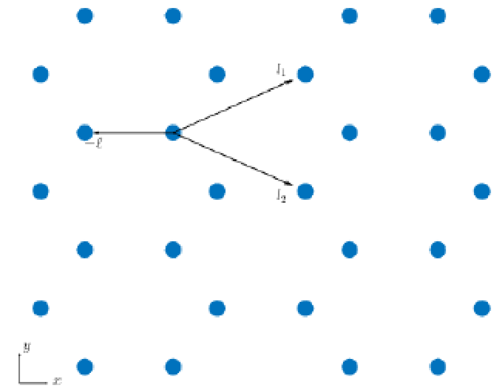


FIG. 5. An illustration of the extended lattice $\mathfrak{D}$.

Momentum space

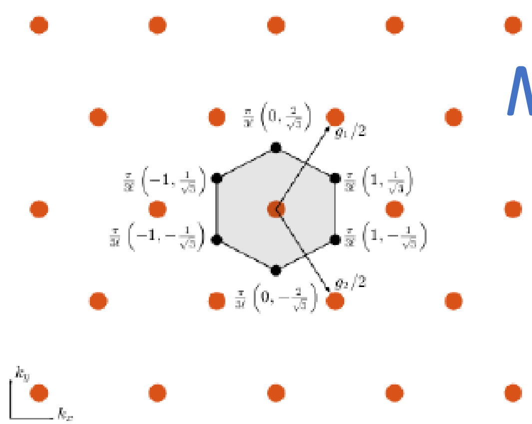


FIG. 3. The first Brillouin zone and the reciprocal lattice of $\mathscr{O}$.

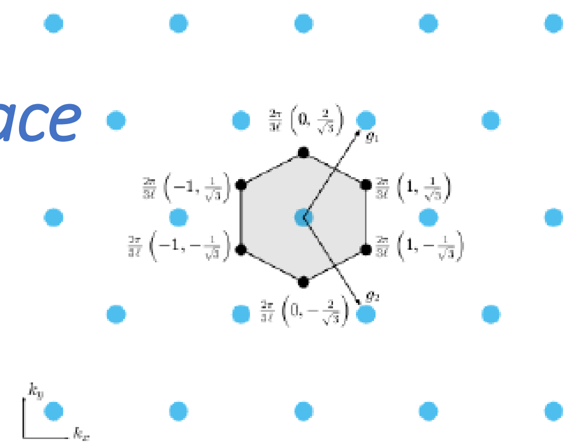


FIG. 6. The first Brillouin zone and the reciprocal lattice of $\mathfrak{D}$.

Precise Wigner – Weyl calculus. INFINITE HONEYCOMB lattice N → infinity

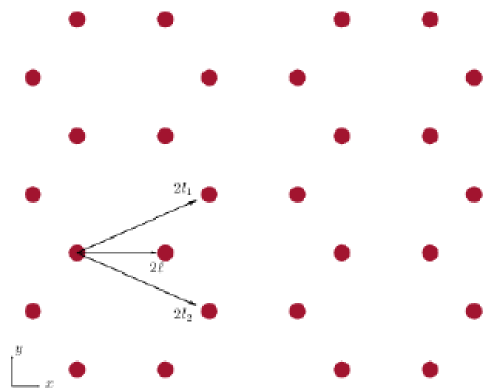


FIG. 2. An illustration of the physical lattice $\mathscr{O}$.

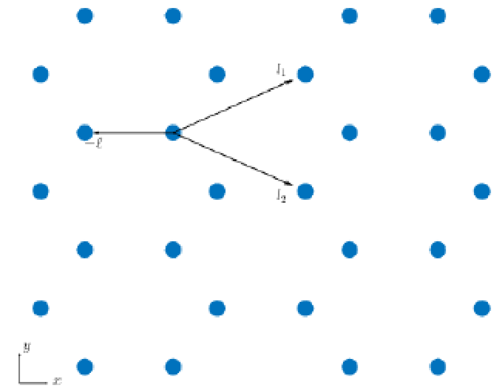


FIG. 5. An illustration of the extended lattice $\mathfrak{D}$.

Weyl symbol of operator

$$A_W(x,p) \equiv \int_{\mathscr{M}} d^2q e^{2ixq} \langle p+q|\hat{A}|p-q\rangle \times \left(1 + e^{-2il_1 q} + e^{-2il_2 q} + e^{-2i(l_1+l_2)q}\right)$$

Properties of Weyl symbol N → infinity

$$\mathrm{Tr}\,\hat{A} = \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathfrak{M}|} A_W(x,p)$$

$$\mathrm{Tr}\hat{A}\hat{B} = \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathfrak{M}|} A_W(x,p) B_W(x,p)$$

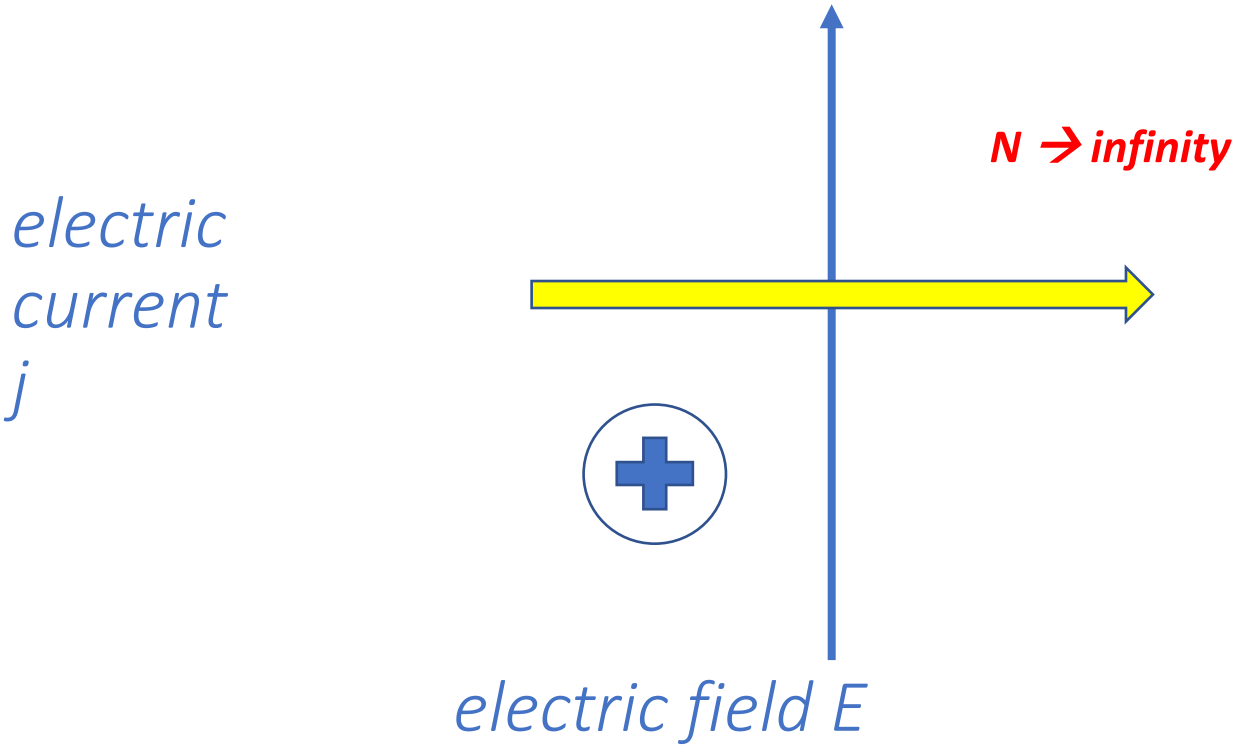
$$(\hat{A}\hat{B})_W(x,p)\Big|_{p\in\mathcal{M},x\in\mathfrak{D}} = A_W(p,q) e^{\frac{i}{2}(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q})} B_W(p,q)$$

$$1_W(x,p)\Big|_{p\in\mathcal{M},x\in\mathfrak{D}} = 1$$

Applications: QHE

$$\bar{\sigma}^{ij} = \frac{\mathcal{N}}{2\pi}\epsilon^{ij}$$

$$\mathcal{N} = \frac{1}{24\,\pi^2}\epsilon^{\mu\nu\rho}\frac{1}{|\mathfrak{D}|}\int dP^0 \int_{\mathscr{M}} d^2\vec{P} \sum_{x\in\mathfrak{D}} \mathrm{tr}\left(\partial_{\Pi^\mu}\hat{Q}_W^{\mathrm{M}} \star \hat{G}_W^{\mathrm{M}} \star \partial_{\Pi^\nu}\hat{Q}_W^{\mathrm{M}} \star \hat{G}_W^{\mathrm{M}} \star \partial_{\Pi^\rho}\hat{Q}_W^{\mathrm{M}}\big|_\star \hat{G}_W^{\mathrm{M}}\right)$$

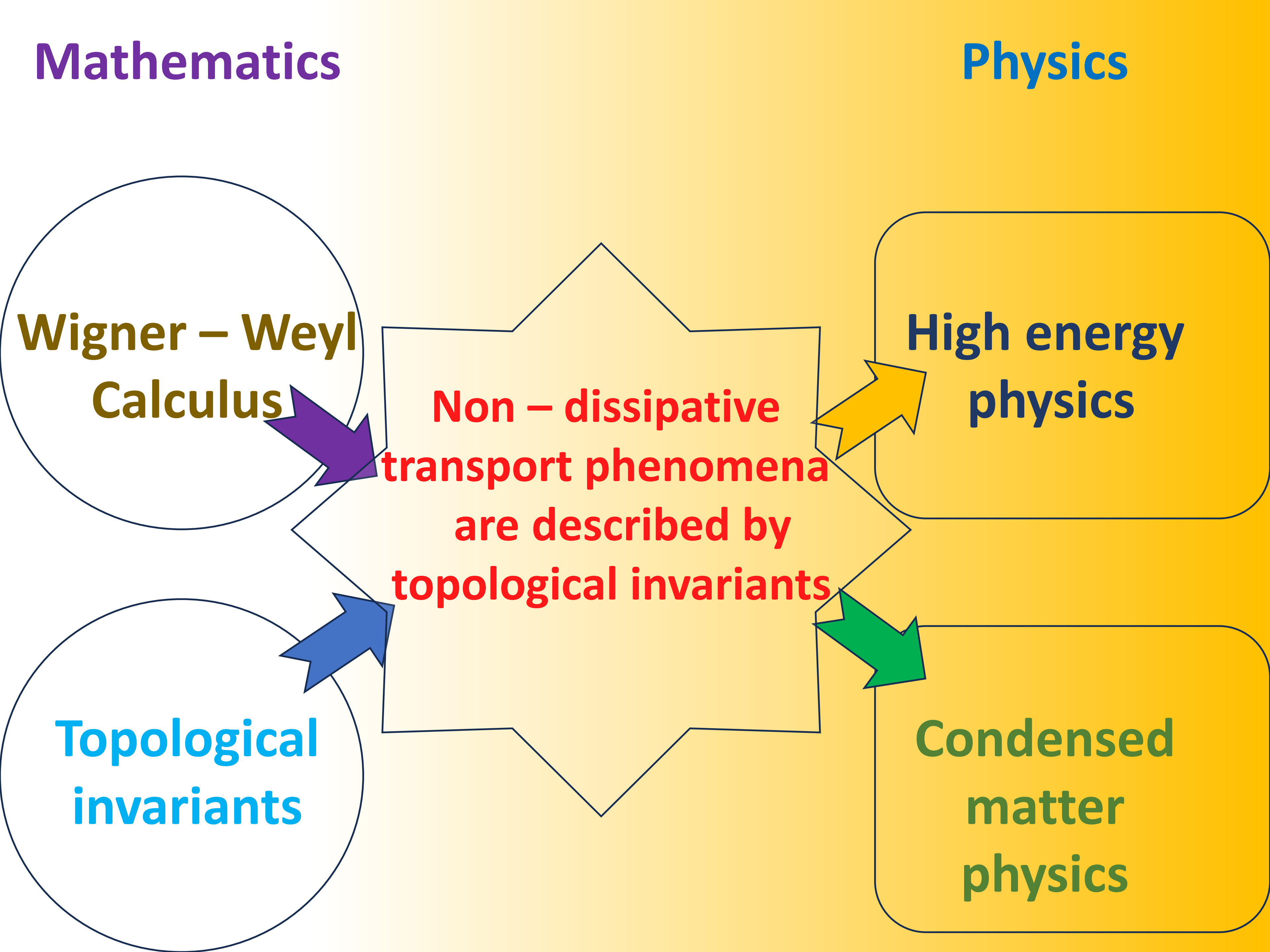


R. Chobanyan, M.A. Zubkov arXiv preprint arXiv:2302.00723
Symmetry 2024, 16(8), 1081

We can use the precise Wigner – Weyl calculus dealing with ***any lattice regularized continuum quantum field theory***

and dealing with the lattice models of solid state physics ***if the external magnetic field strength is of the order of 10 000 Tesla (unphysical!)*** *while wavelength of external electromagnetic field is of the order of* ***1 nanometer***

Which is more important, we can use this formalism for artificial lattices, when magnetic flux through the EFFECTIVE lattice cell is compared to 1



Mathematics
Physics
Wigner – Weyl Calculus
Topological invariants
Non – dissipative transport phenomena are described by topological invariants
High energy physics
Condensed matter physics

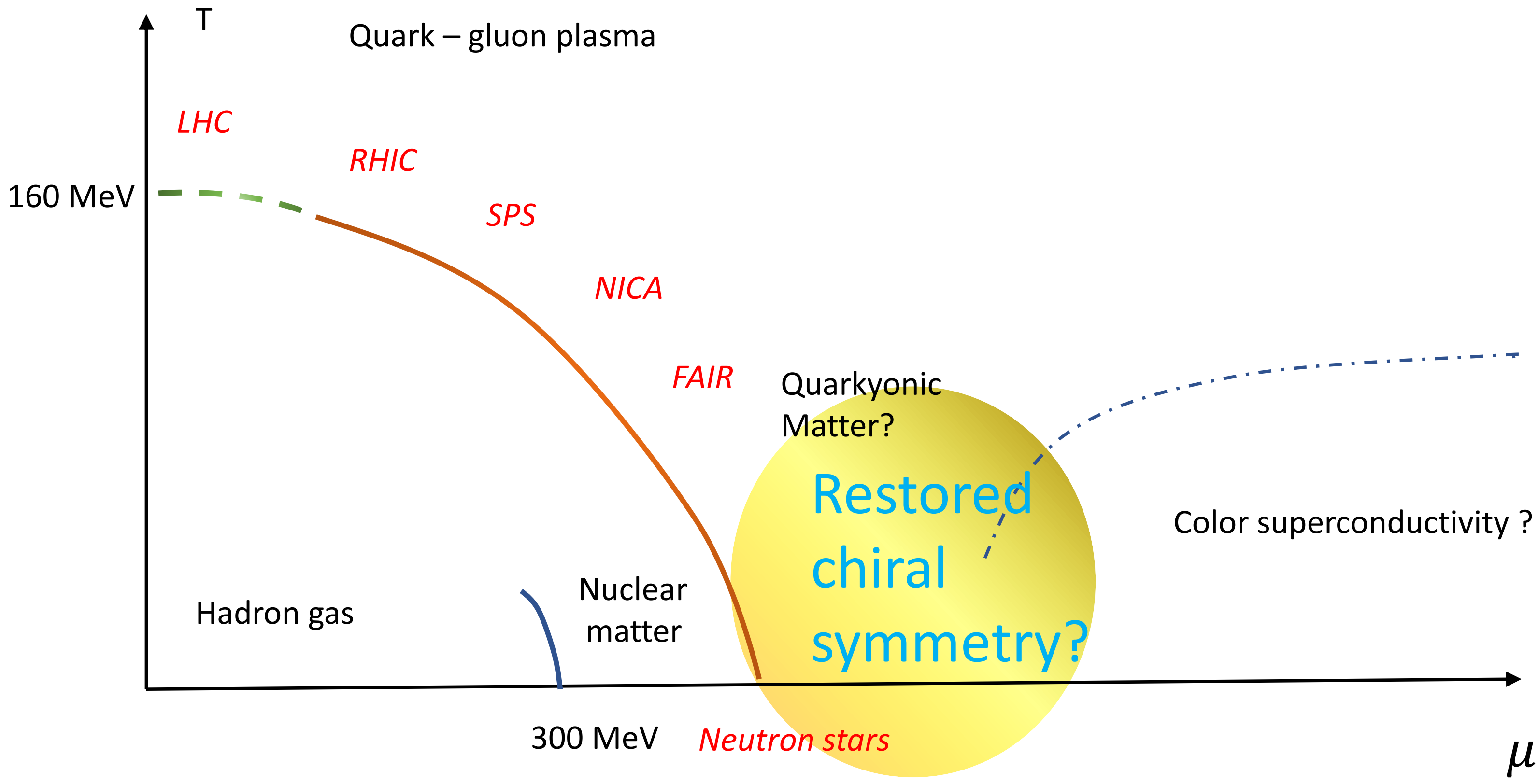
Non – dissipative transport in quark matter

Chiral separation effect (CSE): Axial current in the presence of magnetic field

Chiral vortical effect (CVE): Axial current in the presence of rotation

Chiral magnetic effect (CME): Vector current in the presence of magnetic field

And chiral disbalance



Non – dissipative transport in condensed matter

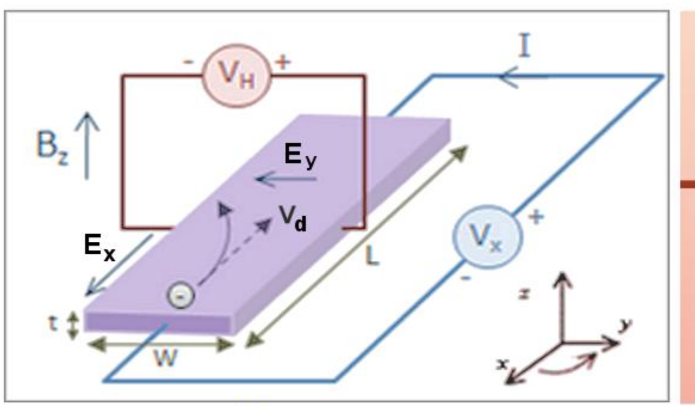
Quantum Hall effect (QHE): Electric current orthogonal to electric field

Chiral separation effect (CSE): Axial current in the presence of magnetic field

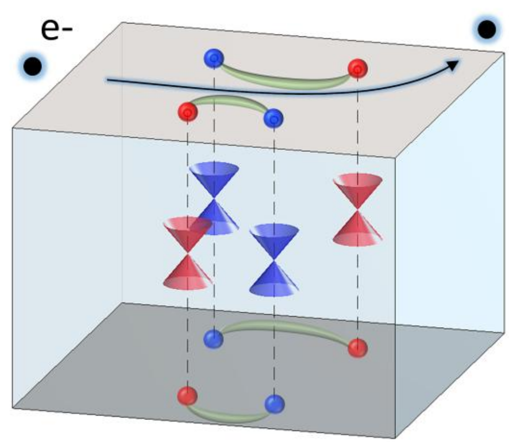
Chiral vortical effect (CVE): Axial current in the presence of rotation

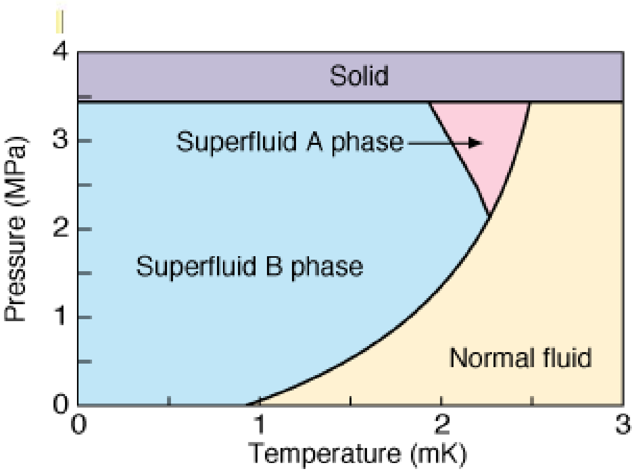
Chiral magnetic effect (CME): Vector current in the presence of magnetic field

And chiral disbalance



(a) Hall Effect





2d materials: QHE 3d Weyl semimetals: CSE, CME, QHE He3-A superfluid:CVE

Conclusions

- Wigner – Weyl calculus allows to represent in compact form the conductivities of non – dissipative transport phenomena **<u>in non – uniform systems</u>**.
- In equilibrium systems these conductivities are given by topological invariants composed of the Wigner transformed two-point Green functions. This expression is not renormalized by interactions (perturbatively). We considered this for the cases of CME and CSE. (The case of CME is marginal: the CME conductivity is zero.)

Conclusions

- We consider the non – Abelian versions of quantum Hall effect and chiral separation effect. Their conductivities are the same as for their Abelian versions.
- Chiral anomaly is equal to the product of the topological invariant responsible for the CSE and the number of instantons. This may have experimental consequences if Dirac operator is not linear in momentum in certain circumstances.

Conclusions

- Out of equilibrium the CME is back if chiral chemical potential depends on time and if the corresponding frequency tends to zero (i.e. the system is approaching to equilibrium).
- Precise Wigner – Weyl calculus is built for the lattice models, which allows us to investigate the lattice regularized QFT precisely. So far the application of this technique was proposed to the consideration of the QHE for the condensed matter systems with artificial lattices (when magnetic flux through the lattice cell becomes large– these are the systems that possess Hofstadter butterfly.

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