A note on the theory of well orders

Emil Jeřábek

Institute of Mathematics, Czech Academy of Sciences
Žitná 25, 115 67 Praha 1, Czech Republic, email: jerabek@math.cas.cz

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Abstract

We give a simple proof that the first-order theory of well orders is axiomatized by transfinite induction, and that it is decidable.

The first-order theory of the class $\mathcal{WO}$ of well-ordered sets $\langle L, <\rangle$ was developed by Tarski and Mostowski, and an in-depth analysis was finally published by Doner, Mostowski, and Tarski [1]: among other results, they provided an explicit axiomatization for the theory, and proved it decidable. Their main technical tool is a syntactic elimination of quantifiers, which however takes some work to establish, as various somewhat nontrivial properties of Cantor normal forms are definable in the theory after all. Alternatively, by way of hammering nails with a nuke, the decidability of $\mathrm{Th}(\mathcal{WO})$ follows from an interpretation of the MSO theory of linear orders in the MSO theory of two successors (S2S), which is decidable by a well-known difficult result of Rabin [4]. Our goal is to point out that basic properties of $\mathrm{Th}(\mathcal{WO})$ can be proved easily using ideas from Läuchli and Leonard's [3] proof of the decidability of the theory of linear orders.

Let $\mathcal{L}_<$ denote the set of sentences in the language $\{<\}$. The theory of (strict) linear orders is denoted LO; the $\mathcal{L}_<$-theory TI extends LO with the transfinite induction schema

$$\forall x\,\bigl(\forall y\,(y < x \to \varphi(y)) \to \varphi(x)\bigr) \to \forall x\,\varphi(x)$$

for all formulas φ (that may in principle include other free variables as parameters, though the parameter-free version is sufficient for our purposes). We will generally denote a linearly ordered set $\langle L, <\rangle$ as just L. Given linearly ordered sets I and L_i for $i \in I$, let $\sum_{i\in I} L_i$ denote the ordered sum with domain $\{\langle i, x\rangle : i \in I, x \in L_i\}$ and lexicographic order

$$\langle i, x\rangle < \langle j, y\rangle \iff i < j \text{ or } (i = j \text{ and } x < y).$$

We write $L \equiv_k L'$ if $L \vDash \varphi \iff L' \vDash \varphi$ for all $\varphi \in \mathcal{L}_<$ of quantifier rank $\mathrm{rk}(\varphi) \le k$. It follows from the basic theory of Ehrenfeucht–Fraïssé games (see [2, 3]) that for each $k < \omega$, $\equiv_k$ has only finitely many equivalence classes as there are only finitely many formulas of rank $\le k$ up to logical equivalence, and that $\equiv_k$ preserves ordered sums:

Lemma 1 *If $L_i \equiv_k L'_i$ for each $i \in I$, then $\sum_{i\in I} L_i \equiv_k \sum_{i\in I} L'_i$.* $\square$

Theorem 2 *The following are equivalent for all $\varphi \in \mathcal{L}_<$:*

(i) $\mathcal{WO} \vDash \varphi$.

(ii) $\alpha \vDash \varphi$ *for all* $\alpha < \omega^\omega$.

(iii) $\mathsf{TI} \vdash \varphi$.

Proof: Clearly, (iii) $\to$ (i) $\to$ (ii). For (ii) $\to$ (iii), if $\mathsf{TI} \nvdash \varphi$, let L be a countable model of $\mathsf{TI} + \neg\varphi$, and $k = \mathrm{rk}(\varphi)$; it suffices to show that there exists $\alpha < \omega^\omega$ such that $L \equiv_k \alpha$. Put

$$S = \{c \in L : \forall a < b \leq c\, \exists \alpha < \omega^\omega\, [a, b) \equiv_k \alpha\}.$$

While the definition speaks of half-open intervals $[a, b)$, the conclusion also holds for $[a, b]$: if $[a, b) \equiv_k \alpha$, then $[a, b] \equiv_k \alpha + 1$ by Lemma 1. Clearly, S is an initial segment of L, and $0 \in S$, where $0 = \min(L)$ (which exists by $L \vDash \mathsf{TI}$).

Claim 2.1 *S is definable in L.*

Proof: Since there are only finitely many formulas of rank $\leq k$ up to equivalence, we can form $\theta_k = \bigwedge\{\theta \in \mathcal{L}_< : \mathrm{rk}(\theta) \leq k, \forall \alpha < \omega^\omega\, \alpha \vDash \theta\}$. Then for any linear order L', $L' \equiv_k \alpha$ for some $\alpha < \omega^\omega$ iff $L' \vDash \theta_k$. In particular, $c \in S$ iff $L \vDash \forall x, y\, (x < y \leq c \to \theta_k^{[x,y)})$, where $\theta_k^{[x,y)}$ denotes θ_k with quantifiers relativized to $[x, y)$. $\square$ (Claim 2.1)

First, assume that S has a largest element, say m. If $S = L$, then $L = [0, m] \equiv_k \alpha$ for some $\alpha < \omega^\omega$, and we are done. Otherwise, we will derive a contradiction by showing that the successor of m (which exists by TI), denoted c, belongs to S. Indeed, if $a < b \leq c$, then either $b \leq m$, in which case $[a, b) \equiv_k \alpha$ for some $\alpha < \omega^\omega$ as $m \in S$, or $b = c$, in which case $[a, b) = [a, m] \equiv_k \alpha$ for some $\alpha < \omega^\omega$ as well.

Thus, we may assume that S has no largest element. Put $S_{\geq a} = \{b \in S : b \geq a\}$.

Claim 2.2 *For every $a \in S$, there is $\alpha < \omega^\omega$ such that $S_{\geq a} \equiv_k \alpha$.*

Proof: We use the idea of [3, Lem. 8]. Let $a < a_0 < a_1 < a_2 < \cdots$ be a cofinal sequence in S. For each $n < m < \omega$, let $t(\{n, m\}) = \min\{\alpha < \omega^\omega : [a_n, a_m) \equiv_k \alpha\}$. Since $\equiv_k$ has only finitely many equivalence classes, t is a colouring of pairs of natural numbers by finitely many colours; by Ramsey's theorem, there is $\beta < \omega^\omega$ and an infinite $H \subseteq \omega$ such that $t(\{n, m\}) = \beta$ for all $n, m \in H$, $n \neq m$. Let $\{b_n : n < \omega\}$ be the increasing enumeration of $\{a_n : n \in H\}$, and $\alpha < \omega^\omega$ be such that $[a, b_0) \equiv_k \alpha$. Then $S_{\geq a} = [a, b_0) + \sum_{n<\omega}[b_n, b_{n+1}) \equiv_k \alpha + \beta \cdot \omega < \omega^\omega$ by Lemma 1. $\square$ (Claim 2.2)

Now, if $S = L$, then $L = S_{\geq 0} \equiv_k \alpha$ for some $\alpha < \omega^\omega$ by Claim 2.2. Otherwise, there exists $c = \min(L \smallsetminus S)$ by Claim 2.1 and $L \vDash \mathsf{TI}$. We again derive a contradiction by showing $c \in S$: if $a < b \leq c$, then either $b < c$ and $[a, b) \equiv_k \alpha$ for some $\alpha < \omega^\omega$ as $b \in S$, or $b = c$ and $[a, b) = S_{\geq a} \equiv_k \alpha$ for some $\alpha < \omega^\omega$ by Claim 2.2. $\square$

We have so far not actually used any results of Läuchli and Leonard [3], only their methods. But we will do so now, in order to get decidability. Let $\mathbf{W}$ denote the set of closed terms in a language with a constant 1, a binary function $x+y$, and a unary function $x \cdot \omega$. Elements of $\mathbf{W}$ are interpreted as names for ordinals; by considering the Cantor normal form, we see that every nonzero $\alpha < \omega^\omega$ is denoted by a term from $\mathbf{W}$. The following is a special case of [3, Thm. 1] (they employ a larger set of terms, but the extra operations are not relevant for well orders); we outline the main idea as the proof can be somewhat simplified in the well ordered case:

Theorem 3 *The relation* $\{\langle \alpha, \varphi \rangle \in \mathbf{W} \times \mathcal{L}_< : \alpha \vDash \varphi\}$ *is decidable.*

Proof sketch: By induction on the complexity of a term $\alpha \in \mathbf{W}$, we describe $\mathrm{Th}(\alpha)$ by means of an r.e. theory T_α. Clearly, $\mathrm{Th}(1) \equiv \forall x, y\,(x = y \wedge x \not< y) =: T_1$. We have $\mathrm{Th}(\alpha + \beta) \equiv \mathsf{LO} + \{\exists x\,(\varphi^{<x} \wedge \psi^{\geq x}) : \alpha \vDash \varphi, \beta \vDash \psi\} =: T_{\alpha+\beta}$, where $\varphi^{<x}$ denotes the sentence φ with all quantifiers relativized to $(-\infty, x) = \{y : y < x\}$, and similarly for $\psi^{\geq x}$: in particular, any model of $T_{\alpha+\beta}$ has an elementary extension L with an element $a \in L$ such that $(-\infty, a) \equiv \alpha$ and $[a, \infty) \equiv \beta$, which implies $L \equiv \alpha + \beta$ by Lemma 1.

Finally, $\mathrm{Th}(\alpha \cdot \omega) = T_{\alpha \cdot \omega} \restriction \mathcal{L}_<$, where $T_{\alpha\cdot\omega}$ is a theory with an extra predicate $P(x)$ and axioms ensuring that $\langle L, <, P \rangle \vDash T_{\alpha \cdot \omega}$ is a linear order with a least element 0, every $x \in L$ belongs to $[a, a^+)$ for some $a, a^+ \in P$ such that a^+ is a successor of a in P (i.e., $(a, a^+) \cap P = \varnothing$), every nonzero $a \in P$ has a predecessor $b \in P$ (i.e., $b^+ = a$), and for each $a \in P$, $[a, a^+) \equiv \alpha$. These axioms imply that $\langle P, < \rangle$ is a discrete order with a least element and no largest element, i.e., $\langle P, < \rangle \equiv \omega$, and we have $L = \sum_{a \in P} [a, a^+) \equiv \sum_{a \in P} \alpha \equiv \alpha \cdot \omega$ by a variant of Lemma 1. Conversely, $\alpha \cdot \omega$ expands to a model of $T_{\alpha\cdot\omega}$.

One can set up a computable notion of "proof sequences" $\langle \alpha_0, \varphi_0 \rangle, \dots, \langle \alpha_n, \varphi_n \rangle$ witnessing that $T_{\alpha_i} \vdash \varphi_i$ (see [3, Lem. 6] for details). This shows $\{\langle \alpha, \varphi \rangle \in \mathbf{W} \times \mathcal{L}_< : \alpha \vDash \varphi\}$ is r.e., hence, in view of $\alpha \nvDash \varphi \iff \alpha \vDash \neg\varphi$, it is decidable. $\square$

Corollary 4 *The theory* $\mathrm{Th}(\mathcal{WO}) = \mathsf{TI}$ *is decidable.*

Proof: $\{\varphi \in \mathcal{L}_< : \mathsf{TI} \vdash \varphi\}$ is recursively enumerable as TI is recursively axiomatized; by Theorems 2 and 3, $\{\varphi \in \mathcal{L}_< : \mathsf{TI} \nvdash \varphi\} = \{\varphi \in \mathcal{L}_< : \exists \alpha \in \mathbf{W}\, \alpha \vDash \neg\varphi\}$ is also recursively enumerable. $\square$

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References

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