Elementary analytic functions in VTC^0

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Outline

1. **$\mathbf{TC}^0$ and** VTC^0
2. **Analytic functions in** VTC^0
3. **Construction of** $\exp$
4. **Construction of** $\log$
5. **Applications**
6. **Exponential integer parts**

$\mathbf{TC}^0$ and VTC^0

The class $\mathbf{TC}^0$

$$\mathbf{AC}^0 \subseteq \mathbf{ACC}^0 \subseteq \mathbf{TC}^0 \subseteq \mathbf{NC}^1 \subseteq \mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{AC}^1 \subseteq \cdots \subseteq \mathbf{P}$$

$\mathbf{TC}^0$ = dlogtime-uniform $O(1)$-depth $n^{O(1)}$-size unbounded fan-in circuits with threshold gates

= **FOM**-definable on finite structures representing strings (first-order logic with majority quantifiers)

= $O(\log n)$ time, $O(1)$ thresholds on a threshold Turing machine

= Constable's $\mathcal{K}$: closure of $+, -, \cdot, /$ under substitution and polynomially bounded $\sum$, $\prod$

$\mathbf{TC}^0$ and arithmetic operations

For integers given in binary:

- $+$ and $\leq$ are in $\mathbf{AC}^0 \subseteq \mathbf{TC}^0$
- $\times$ is in $\mathbf{TC}^0$ ($\mathbf{TC}^0$-complete under $\mathbf{AC}^0$ reductions)

$\mathbf{TC}^0$ can also do:

- iterated addition $\sum_{i<n} X_i$
- integer division and iterated multiplication [HAB'02]
- the corresponding operations on $\mathbb{Q}$, $\mathbb{Q}(\alpha)$, ...
- approximate functions given by nice power series:
  - $\sin X$, $\log X$, $\sqrt[k]{X}$, ...
- sorting, ...

The theory VTC^0

- Zambella-style two-sorted bounded arithmetic
  - unary (auxiliary) integers with $0, 1, +, \cdot, \leq$
  - finite sets = binary integers = binary strings
- Noteworthy axioms:
  - Σ^B_0-comprehension (Σ^B_0 = bounded, w/o SO q'fiers)
  - every set has a counting function
- Correspondence to $\mathbf{TC}^0$:
  - provably total computable (i.e., $\exists\Sigma^B_0$-definable) functions are exactly the $\mathbf{TC}^0$ functions
  - has induction, minimization, ... for $\mathbf{TC}^0$ predicates
- Equivalent (RSUV-isomorphic) to Δ^b_1-CR of [JP'00]
  - Buss-style one-sorted bounded arithmetic
  - Open LIND, Δ^b_1 bit-comprehension rule

Binary integer arithmetic in VTC^0

Basic integer arithmetic in VTC^0:

- can define $+, \cdot, \leq$ on binary integers
- proves integers form a discretely ordered ring (*DOR*)

More sophisticated:

- [J'22] iterated multiplication and division
  - formalize a variant of the [HAB'02] algorithm
- [J'15] open induction in $\langle +, \cdot, < \rangle$ (IOpen), Σ^b_0-minimization and Σ^b_0-induction in Buss's language
  - $\approx$ **TC**^{0} constant-degree polynomial root approximation

Analytic functions in VTC^0

Elementary analytic functions

Recall: $\mathbf{TC}^0$ can compute approximations of analytic functions whose power series have $\mathbf{TC}^0$-computable coefficients

Question: Can VTC^0 prove their basic properties?

There's a plethora of such functions $\implies$ let's start small:

Elementary analytic functions (real and complex)

- exp, log
- trigonometric: sin, cos, tan, cot, sec, csc
- inverse trig.: arcsin, arccos, arctan, arccot, arcsec, arccsc
- hyperbolic: sinh, cosh, tanh, coth, sech, csch
- inverse hyp.: arsinh, arcosh, artanh, arcoth, arsech, arcsch

All definable in terms of complex exp and log

VTC^0 setup

Working with rational approximations only is quite tiresome:

- statements of theorems messy
- keep track of approximation parameters everywhere
  $\implies$ ε-δ analysis at its worst

Solution: work with larger structures where analytic functions can be defined as bona fide functions

Given a model $\mathfrak{M} \vDash \mathrm{VTC}^0$, form

- discretely ordered ring $\mathbf{Z}^{\mathfrak{M}}$ (binary integers)
- fraction field $\mathbf{Q}^{\mathfrak{M}}$
- completion $\mathbf{R}^{\mathfrak{M}}$ (real-closed field [J'15])
- algebraic closure $\mathbf{C}^{\mathfrak{M}} = \mathbf{R}^{\mathfrak{M}}(i)$ (still complete)

Completions of ordered fields

Let F be an ordered field (OF)

- F is complete if it is not dense in any proper extension OF
- completion: a complete OF $\hat{F}$ s.t. $F \subseteq \hat{F}$ is dense
- every F has a unique completion (up to isomorphism)

More explicit description:

- cut in F: $\langle A, B \rangle$ s.t. $F = A \cup B$, $\neg\exists \max A$, $\inf\{b - a : b \in B, a \in A\} = 0$
- F complete $\iff$ all cuts are filled ($\exists \min B$)
- $\hat{F}$ = the set of all cuts in F with obvious structure

Topological description

Every OF F carries interval topology
$\Longrightarrow$ topological field $\Longrightarrow$ uniform space

- complete if every Cauchy net converges
- every uniform space S has a unique completion: complete space $\hat{S}$ s.t. $S \subseteq \hat{S}$ dense
- T complete $\Longrightarrow$ every uniformly continuous function $S \to T$ uniquely extends to a uniformly continuous function $\hat{S} \to T$
- topological completion of an OF F has a canonical structure of OF $\hat{F} \supseteq F$, coincides with OF completion

VTC^0 setup (cont'd)

$\mathfrak{M} \vDash \mathrm{VTC}^0 \rightsquigarrow \mathbf{Z}^{\mathfrak{M}} \rightsquigarrow \mathbf{Q}^{\mathfrak{M}} \rightsquigarrow \mathbf{R}^{\mathfrak{M}} \rightsquigarrow \mathbf{C}^{\mathfrak{M}}$

A well-behaved (i.e., Cauchy) sequence of approximations in $\mathbf{Q}^{\mathfrak{M}}(i)$ defines an element of $\mathbf{C}^{\mathfrak{M}}$
$\Longrightarrow$ instead of approximations, treat our analytic functions as $f\colon \mathbf{C}^{\mathfrak{M}} \to \mathbf{C}^{\mathfrak{M}}$ (or on a subset)

NB rational approximations still needed:

- translate results back to the language of VTC^0
- use the functions in induction arguments, ...

Further notation: unary integers embed as $\mathbf{L}^{\mathfrak{M}} \subseteq \mathbf{Z}^{\mathfrak{M}}$

$\mathbf{C}^{\mathfrak{M}}_{\mathbf{L}} = \bigl\{z \in \mathbf{C}^{\mathfrak{M}} : \exists n \in \mathbf{L}^{\mathfrak{M}}\ |z| \le n\bigr\}$, $\mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} = \mathbf{R}^{\mathfrak{M}} \cap \mathbf{C}^{\mathfrak{M}}_{\mathbf{L}}$, ...

(will drop the $^{\mathfrak{M}}$ superscripts)

TC^0 approximations

$f\colon D \to \mathbf{C}^{\mathfrak{M}}$, $D \subseteq \mathbf{C}^{\mathfrak{M}}$ s.t. $D \cap \mathbf{Q}^{\mathfrak{M}}(i)$ is dense in D

Approximation of f by TC^0 functions:

Additive: $\mathbf{TC}^0$ function $f_+\colon \mathbf{Q}^{\mathfrak{M}}(i) \times \mathbf{L}^{\mathfrak{M}} \to \mathbf{Q}^{\mathfrak{M}}(i)$

$$\bigl|f_+(z,n) - f(z)\bigr| \le 2^{-n} \qquad \forall n \in \mathbf{L}^{\mathfrak{M}}, z \in D \cap \mathbf{Q}^{\mathfrak{M}}(i)$$

Multiplicative: $\mathbf{TC}^0$ function $f_\times\colon \mathbf{Q}^{\mathfrak{M}}(i) \times \mathbf{L}^{\mathfrak{M}} \to \mathbf{Q}^{\mathfrak{M}}(i)$

$$\bigl|f_\times(z,n) - f(z)\bigr| \le 2^{-n}|f(z)| \quad \forall n \in \mathbf{L}^{\mathfrak{M}}, z \in D \cap \mathbf{Q}^{\mathfrak{M}}(i)$$

In other words:

$$f(z) = 0 \implies f_\times(z,n) = 0$$

$$f(z) \ne 0 \implies \left|\frac{f_\times(z,n)}{f(z)} - 1\right| \le 2^{-n}$$

Additive vs. multiplicative approximation

For any $f\colon D \to \mathbf{C}^{\mathfrak{M}}$, $D \subseteq \mathbf{C}^{\mathfrak{M}}$, the following are equivalent:

- f has a multiplicative $\mathbf{TC}^0$ approximation
- f has an additive $\mathbf{TC}^0$ approximation, and
  $\exists$ $\mathbf{TC}^0$ function $h\colon \mathbf{Q}^{\mathfrak{M}}(i) \to \mathbf{L}^{\mathfrak{M}}$ s.t.

$$f(z) \neq 0 \implies |f(z)| \geq 2^{-h(z)} \quad \forall z \in D \cap \mathbf{Q}^{\mathfrak{M}}(i)$$

(bound $f(z)$ away from 0)

Main results

We can define $\pi \in \mathbf{R}^{\mathfrak{M}}$,

$$\exp\colon \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} + i\mathbf{R}^{\mathfrak{M}} \to \mathbf{C}^{\mathfrak{M}}_{\neq 0},$$
$$\log\colon \mathbf{C}^{\mathfrak{M}}_{\neq 0} \to \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} + i(-\pi, \pi]$$

such that

- $\exp(z + w) = \exp z \exp w$
- $\exp$ is $2\pi i$-periodic
- $\exp \log z = z$
- $\log \exp z = z$ for $z \in \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} + i(-\pi, \pi]$
- $\exp \restriction \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}}$ increasing bijection $\mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} \to \mathbf{R}^{\mathfrak{M}}_{>0}$, convex
- for small z: $\exp z = 1 + z + O(z^2)$, $\log(1 + z) = z + O(z^2)$
- suitable additive and multiplicative $\mathbf{TC}^0$ approximations

Outline of the arguments

- Define exp: $\mathbf{C}^{\mathfrak{M}}_{\mathbf{L}} \to \mathbf{C}^{\mathfrak{M}}$ using $\sum_n \frac{z^n}{n!}$
  show $\exp(z_0 + z_1) = \exp z_0 \exp z_1$
- Define log on a nbh of 1 using $-\sum_n \frac{(1-z)^n}{n}$
  show $\log(z_0 z_1) = \log z_0 + \log z_1$ for z_j close enough to 1
- Extend log
  - to $\mathbf{R}^{\mathfrak{M}}_{>0}$ using $2^n \colon \mathbf{L}^{\mathfrak{M}} \to \mathbf{Z}^{\mathfrak{M}}$
  - to an angular sector by combining the two
  - to $\mathbf{C}^{\mathfrak{M}}_{\neq 0}$ using $8 \log \sqrt[8]{z}$
- $\log\exp(z_0 + z_1) = \log\exp z_0 + \log\exp z_1$ when $|\mathrm{Im}\, z_j|$ small
  $\implies$ $\log\exp z = z$ when $|\mathrm{Im}\, z|$ small
  $\implies$ $\exp\log z = z$ using injectivity of log
- exp is $2\pi i$-periodic for $\pi := \mathrm{Im}\log(-1)$
  $\implies$ extend exp to $\mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} + i\mathbf{R}^{\mathfrak{M}}$

Construction of $\exp$

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Power series

- define $e\colon \mathbf{Q}(i) \times \mathbf{L} \to \mathbf{Q}(i)$ by

$$e(z,n) = \sum_{j<n} \frac{z^j}{j!}$$

- Cauchy for fixed $z \in \mathbf{Q}_{\mathbf{L}}(i) \implies$ define $\exp\colon \mathbf{Q}_{\mathbf{L}}(i) \to \mathbf{C}$,

$$\exp z = \lim_{\mathbf{L} \ni n \to \infty} e(z,n)$$

- uniformly continuous on $\overline{D}_r(0) = \{z : |z| \le r\}$, $r \in \mathbf{L}$ $\implies$ unique continuous extension $\exp\colon \mathbf{C}_{\mathbf{L}} \to \mathbf{C}$

Homomorphism identity

- binomial identity $\frac{(z+w)^l}{l!} = \sum_{j+k=l} \frac{z^j w^k}{j!\,k!} \implies$

$$e(z+w,2n) - e(z,n)e(w,n) = \sum_{\substack{j+k<2n \\ \max\{j,k\}\ge n}} \frac{z^j w^k}{j!\,k!}$$
$$= O(2^{-n}\exp r)$$

for $z,w \in \overline{D}_r(0) \cap \mathbf{Q}(i)$, $r \in \mathbf{L}$, $n \ge 8r$

- taking limits and using continuity,

$$\exp(z+w) = \exp z \exp w \qquad \forall z,w \in \mathbf{C}_{\mathbf{L}}$$

Checkpoint

Can prove at this point:

- exp homomorphism $\langle \mathbf{C_L}, +, 0, - \rangle \to \langle \mathbf{C}_{\neq 0}, \cdot, 1, ^{-1} \rangle$
- $\exp \restriction \mathbf{R_L}$ homomorphism $\langle \mathbf{R_L}, +, 0, -, < \rangle \to \langle \mathbf{R}_{>0}, \cdot, 1, ^{-1}, < \rangle$
- $\exp \overline{z} = \overline{\exp z}$, $|\exp z| = \exp \operatorname{Re} z$
- $|z| \le \frac{3}{2} \implies \bigl|\exp z - (1 + z)\bigr| \le |z|^2$
- $\exp x \ge 1 + x$ for $x \in \mathbf{R_L}$, $\exp \restriction \mathbf{R_L}$ is convex

Still missing:

- exp and $\exp \restriction \mathbf{R_L}$ are surjective ($\exp \restriction \mathbf{R_L}$ isomorphism)
- $\exists \pi \ \exp(\pi i) = -1 \implies$ exp $2\pi i$-periodic
  $\implies$ extend exp to $\mathbf{R_L} + i\mathbf{R}$

Need to construct log, prove $\exp \log z = z$

Construction of log

Overview of the construction

$\log$ not entire (branching singularity at 0) $\Longrightarrow$ trouble

- power series only works in a neighbourhood of 1
- $\log(zw) = \log z + \log w$ does not really hold

Construction in several stages:

- power series $\Longrightarrow$ $\log_D$ on a disk around 1
- combine with $2^n \colon \mathbf{L} \to \mathbf{Z}$ $\Longrightarrow$ $\log_{\mathbf{R}}$ on $\mathbf{R}_{>0}$
- combine $\log_D$ and $\log_{\mathbf{R}}$ $\Longrightarrow$ $\log_S$ on an angular sector
- use $\sqrt{z}$ to increase the angle $\Longrightarrow$ $\log$ on $\mathbf{C}_{\neq 0}$

Most important arguments:

- $\log(z_0 z_1) = \log z_0 + \log z_1$ when $\operatorname{Re} z_j > 0$
- $\log\exp(z_0 + z_1) = \log\exp z_0 + \log\exp z_1$ when $|\operatorname{Im} z_j| < 1$

  $\Longrightarrow$ $\log\exp z = z$ when $|\operatorname{Im} z| < 1$

  $\Longrightarrow$ $\exp\log z = z$ using injectivity of log

Power series

- define $\lambda\colon \mathbf{Q}(i) \times \mathbf{L} \to \mathbf{Q}(i)$ by

$$\lambda(z, n) = \sum_{j=1}^{n} \frac{z^j}{j}$$

- $D^*_r(z_0) = \{z : |z - z_0| <^* r\}$, where $x <^* y \iff x \le y - h^{-1}$ for some $h \in \mathbf{L}$
- λ Cauchy for $z \in D^*_1(0) \implies \Lambda\colon D^*_1(0) \cap \mathbf{Q}_{\mathbf{L}}(i) \to \mathbf{C}$,

$$\Lambda(z) = \lim_{\mathbf{L} \ni n \to \infty} \lambda(z, n)$$

- Λ uniformly continuous on $\overline{D}_{1-h^{-1}}(0)$, $h \in \mathbf{L} \implies -\Lambda(1-z)$ has continuous extension $\log_D\colon D^*_1(1) \to \mathbf{C}$
- $|z| \le \frac{1}{2} \implies \bigl|\log_D(1+z) - z\bigr| \le |z|^2$

Homomorphism identity

Goal: $(1+r)(1+s) <^* 2 \implies$

(HI) $\quad \log_D zw = \log_D z + \log_D w, \quad z \in \overline{D}_r(1), w \in \overline{D}_s(1)$

In particular, (HI) holds for $z, w \in \overline{D}_{2/5}(1)$

This follows from

$$\left|\lambda(z,n)+\lambda(w,n)-\lambda(z+w-zw,n)\right| \le \frac{(r+s+rs)^{n+1}}{(n+1)(1-r-s-rs)}$$

which in turn follows from

$$\lambda(z,n)+\lambda(w,n)-\lambda(z+w-zw,n) = \sum_{\substack{j,k,l \\ j+l,k+l\le n<j+k+l}} \binom{j+k+l}{j,k,l} \frac{(-1)^l z^{j+l} w^{k+l}}{j+k+l}$$

Homomorphism identity (cont'd)

- backwards difference: $(\nabla f)(x) = f(x) - f(x-1)$
- f polynomial of degree $h < n \implies \nabla^n f = 0$
- take $f =$ falling factorial $x^{\underline{h}} = \prod_{j<h}(x-j)$:

$$\sum_{k\le n} \binom{n}{k} (-1)^k (x-k)^{\underline{h}} = 0$$

- VTC^0 proves this for $h < n \in \mathbf{L}$, $x \in \mathbf{Q}$ by induction
- this implies

$$\sum_{\substack{j,k,l \\ 0<j+l,k+l\le n}} \binom{j+k+l}{j,k,l} \frac{(-1)^l z^{j+l} w^{k+l}}{j+k+l} = 0$$

Real logarithm

Define $\log_{\mathbf{R}} \colon \mathbf{R}_{>0} \to \mathbf{R}_{\mathbf{L}}$ by

$$\log_{\mathbf{R}} 2^n x = \log_D x + n\ell_2, \qquad n \in \mathbf{Z}_{\mathbf{L}}, x \in \left(\tfrac{1}{3}, \tfrac{3}{2}\right)^*$$

where $\ell_2 = -\log_D \frac{1}{2} = \Lambda\left(\frac{1}{2}\right)$

- (HI) $\implies$ $\log_D 2x = \log_D x + \ell_2$ for all $x \in \left(\frac{1}{3}, \frac{3}{4}\right)^*$
  $\implies$ $\log_{\mathbf{R}} 2^n x$ independent of the choice of n, x
- $\log_{\mathbf{R}}$ continuous, strictly increasing
  $\because$ $\log_D$ increasing on $\left[\frac{1}{2}, 1\right]$
- $\log_{\mathbf{R}}$ satisfies (HI) for all $x, y \in \mathbf{R}_{>0}$

Logarithm in angular sector

Complex sign: $\operatorname{sgn} z = z/|z|$ $(z \neq 0)$

$S = \{z \in \mathbf{C}_{\neq 0} : |\operatorname{sgn} z - 1| <^* 1\} = \{x + iy : |y| <^* \sqrt{3}x\}$

Define $\log_S \colon S \to \mathbf{C_L}$ by

$$\log_S z = \log_{\mathbf{R}} |z| + \log_D \operatorname{sgn} z$$

- $\log_S$ satisfies (HI) for elements of $\left\{z \in S : \left|\frac{y}{z}\right| \leq \frac{2}{5}\right\}$
- $\log_S$ extends $\log_{\mathbf{R}}$ and $\log_D \restriction \overline{D}_{2/5}(1)$
  (in fact: all of $\log_D$, but not so easy to prove)

Complex square root

$$z = x + iy \implies \sqrt{z} = \sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\,\mathrm{sgn}^+ y$$

$$\text{where } \mathrm{sgn}^+ y = \begin{cases} 1 & \text{if } y \geq 0, \\ -1 & \text{if } y < 0 \end{cases}$$

- $\left(\sqrt{z}\right)^2 = z$, $\mathrm{sgn}^+ \operatorname{Im} \sqrt{z} = \mathrm{sgn}^+ \operatorname{Im} z$
- $z \notin \mathbf{R}_{<0} \implies \sqrt{\overline{z}} = \overline{\sqrt{z}}$, $\sqrt{z^{-1}} = \left(\sqrt{z}\right)^{-1}$
- $\operatorname{Re} z \geq 0$, $\operatorname{Re} w > 0 \implies \sqrt{zw} = \sqrt{z}\sqrt{w}$
- $\mathrm{sgn}^+ \operatorname{Im} zw \in \{\mathrm{sgn}^+ \operatorname{Im} z, \mathrm{sgn}^+ \operatorname{Im} w\}$, $z \notin \mathbf{R}_{<0}$
  $\implies \sqrt{zw} = \sqrt{z}\sqrt{w}$

Manipulating sectors using $\sqrt{z}$

Let $w = \sqrt{z}$:

$$
\begin{aligned}
z \text{ any} &\implies \operatorname{Re} w \geq 0 \\
\operatorname{Re} z \geq 0 &\implies |\operatorname{Im} w| \leq \operatorname{Re} w \\
|\operatorname{Im} z| \leq \operatorname{Re} z &\implies |\operatorname{Im} w| \leq \tfrac{2}{5}|w|
\end{aligned}
$$

Full logarithm

Define log: $\mathbf{C}_{\neq 0} \to \mathbf{C}_{\mathbf{L}}$ by

$$\log z = 8 \log_S \sqrt{\sqrt{\sqrt{z}}}$$

- extends $\log_S \restriction \{x + iy : |y| \le x\}$ (in fact: all of $\log_S$)
- $z \notin \mathbf{R}_{\le 0} \implies \log z^{-1} = -\log z$, $\log \overline{z} = \overline{\log z}$
- $\log z = 2 \log \sqrt{z}$
- log satisfies (HI) if $\operatorname{Re} z \ge 0$, $\operatorname{Re} w > 0$
- also: if $\operatorname{sgn}^+ \operatorname{Im} zw \in \{\operatorname{sgn}^+ \operatorname{Im} z, \operatorname{sgn}^+ \operatorname{Im} w\}$, $z \notin \mathbf{R}_{<0}$

Complex argument function

Define $\arg z = \operatorname{Im} \log z$, $\pi = \arg(-1)$

- $\log z = \log_{\mathbf{R}} |z| + i \arg z$
- $\operatorname{Re} z, \operatorname{Re} w \geq 0 \implies$

$$\arg z < \arg w \iff \operatorname{Im} \operatorname{sgn} z < \operatorname{Im} \operatorname{sgn} w$$

  and similarly for other quadrants
- $\arg z = \arg w \iff \operatorname{sgn} z = \operatorname{sgn} w$

Consequently:

- log is injective
- $\arg z \in (-\pi, \pi]$ and $\log z \in \mathbf{R_L} + i(-\pi, \pi]$
- $\log z + \log w - \log zw \in \{-2\pi i, 0, 2\pi i\}$

Cauchy functional equation

$z \in \mathbf{R_L} + i(-1,1) \implies \operatorname{Re} \exp z > 0$, consequently:

$$\log \exp(z+w) = \log \exp z + \log \exp w, \quad z, w \in \mathbf{R_L} + i(-1,1)$$

Classically: continuous solutions of $f(z+w) = f(z) + f(w)$ are $f(z) = \alpha \operatorname{Re} z + \beta \operatorname{Im} z$

Idea:

- prove $\log \exp 2^{-n} z = 2^{-n} \log \exp z$ by induction on n
- $\log \exp z = z + O(z^2)$ for small $z \implies$ infer $\log \exp z = z$

Problem: Need $\mathbf{TC}^0$ approximations to use induction!

Parametrized approximation

exp grows too fast to be $\mathbf{TC}^0$ approximable on $\mathbf{Q_L}(i)$

Let $f\colon D \to \mathbf{C}$, $D \subseteq \mathbf{C}$ s.t. $D \cap \mathbf{Q}(i)$ is dense in D:

Additive approximation of f parametrized by $r \in \mathbf{L}$ s.t. $P(z, r)$:
$\mathbf{TC}^0$ function $f_+(z, r, n)$ s.t.

$$P(z, r) \implies |f_+(z, r, n) - f(z)| \le 2^{-n}$$

$(z \in D \cap \mathbf{Q}(i),\ r, n \in \mathbf{L})$

Usually: $\forall z \in D\, \exists r \in \mathbf{L}\, P(z, r)$

Parametrized multiplicative approximation similar

$\mathbf{TC}^0$ approximations

Lemma:

- $\exp z$ has multiplicative (and additive) $\mathbf{TC}^0$ approximation parametrized by $r \in \mathbf{L}$ s.t. $|z| \le r$
- $\log z$ has additive $\mathbf{TC}^0$ approximation
- $\log \exp z$ has $\mathbf{TC}^0$ additive approximation for $|\operatorname{Im} z| < 1$, parametrized by $r \in \mathbf{L}$ s.t. $|z| \le r$

Tedious, but unsurprizing:

- approximate $\exp z$, $\log_D z$, and $\sqrt{x}$ by partial sums
- use bounds on moduli of continuity to combine them

exp and log are mutually inverse

$LE(z,r,n)$ approximation of $\log\exp z$ as above:

- for $z \in \mathbf{Q}(i)$, $r,t,n \in \mathbf{L}$ s.t. $|\mathrm{Im}\, z| < 1$, $|z| \le r$, prove

$$\left|LE(2^{-n}z,r,t) - 2^{-n}LE(z,r,t)\right| \le 3\cdot 2^{-n}$$

  by induction on n
- $\log\exp 2^{-n}z = 2^{-n}\log\exp z$ for $z \in \mathbf{R_L} + i(-1,1)$ by continuity
- $\log\exp z = z + O(z^2)$ for small z $\implies$ $\log\exp z = z$ for $z \in \mathbf{R_L} + i(-1,1)$
- extend to $\log\exp z = z$ for $z \in \mathbf{R_L} + i(-\pi,\pi]$
- log injective $\implies$ $\exp\log z = z$ for $z \in \mathbf{C}_{\neq 0}$

Final extension of exp

$\exp nz = (\exp z)^n$ for $z \in \mathbf{C_L}$, $n \in \mathbf{Z_L}$ "by induction on n"
$\implies \exp(z + 2\pi i\, n) = \exp z$

Extend exp to $\mathbf{R}_{\downarrow\mathbf{L}} + i\mathbf{R}$, $\mathbf{R}_{\downarrow\mathbf{L}} = \mathbf{R}_{<0} \cup \mathbf{R_L}$:

$$\begin{aligned} \exp(z + 2\pi i\, n) &= \exp z && z \in \mathbf{C_L}, n \in \mathbf{Z} \\ \exp z &= 0 && \operatorname{Re} z < \mathbf{R_L} \end{aligned}$$

- $\exp(z + w) = \exp z \exp w$ for all z, w
- $\exp z$ has additive $\mathbf{TC}^0$ approximation for $z \in \mathbf{Q}_{\downarrow\mathbf{L}} + i\mathbf{Q}$ parametrized by $r \in \mathbf{L}$ s.t. $\operatorname{Re} z \le r$
- $\exp z$ has multiplicative $\mathbf{TC}^0$ approximation for $z \in \mathbf{Q_L} + i\mathbf{Q}$ parametrized by $r \in \mathbf{L}$ s.t. $|\operatorname{Re} z| \le r$

Summary

For every $\mathfrak{M} \vDash \mathrm{VTC}^0$, we defined $\pi \in \mathbf{R}^{\mathfrak{M}}$,

$$\exp\colon \mathbf{R}^{\mathfrak{M}}_{\downarrow\mathbf{L}} + i\mathbf{R}^{\mathfrak{M}} \to \mathbf{C}^{\mathfrak{M}},$$
$$\log\colon \mathbf{C}^{\mathfrak{M}}_{\neq 0} \to \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} + i(-\pi, \pi]$$

They satisfy (among other properties):

- $\exp(z + w) = \exp z \exp w$
- exp is $2\pi i$-periodic
- $\exp \log z = z$ for $z \in \mathbf{C}^{\mathfrak{M}}_{\neq 0}$
- $\log \exp z = z$ for $z \in \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} + i(-\pi, \pi]$
- $\exp \restriction \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}}$ increasing bijection $\mathbf{R}^{\mathfrak{M}}_{\mathbf{L}} \to \mathbf{R}^{\mathfrak{M}}_{>0}$, convex
- exp is continuous, log is continuous in $\mathbf{C}^{\mathfrak{M}} \smallsetminus (-\infty, 0]$
- for small z: $\exp z = 1 + z + O(z^2)$, $\log(1 + z) = z + O(z^2)$
- suitable additive and multiplicative $\mathbf{TC}^0$ approximations

Applications

Overview

Define

- $z^w = \exp(w \log z)$, $\sqrt[n]{z} = z^{1/n}$
- $\prod_{j<n} z_j$ for a sequence of $z_j \in \mathbf{Q}^{\mathfrak{M}}(i)$ coded in $\mathfrak{M}$
  - $\approx \exp\left(\sum_{j<n} \log z_j\right)$
- trigonometric, inverse trigonometric, hyperbolic, inverse hyperbolic functions

Model-theoretic consequence:

- Every countable model of VTC^0 is an exponential integer part of a real-closed exponential field

Complex powering

Before: well-behaved z^n for $z \in \mathbf{C}_{\neq 0}$, $n \in \mathbf{Z_L}$

- for $z \in \mathbf{Q}(i)$ iterated multiplication, extend by continuity
- $z^0 = 1$, $z^1 = z$, $z^{n+m} = z^n z^m$, $z^{nm} = (z^n)^m$, $(zw)^n = z^n w^n$

Now: define $z^w = \exp(w \log z)$ for $z \in \mathbf{C}_{\neq 0}$, $w \in \mathbf{C_L}$

- agrees with z^n for $w \in \mathbf{Z_L}$
- $z^{w+w'} = z^w z^{w'}$, $z^{-w} = 1/z^w$
- $(zz')^w = z^w z'^w$ if $\arg z + \arg z' \in (-\pi, \pi]$
- $z^{ww'} = (z^w)^{w'}$ if $w \in (-1, 1]$ or if $z \in \mathbf{R}_{>0}$, $w \in \mathbf{R_L}$

In particular: well-behaved $\sqrt[n]{z} = z^{1/n}$ for $n \in \mathbf{L}_{>0}$

Iterated multiplication

Before: $\prod_{j<n} x_j$ for a sequence $\langle x_j : j < n \rangle$ of $x_j \in \mathbf{Q}$

Also: $(x + iy)^n = \sum_{m \le n} \binom{n}{j} x^m (iy)^{n-m}$

But: ~~$\prod_{j<n}(x_j + iy_j) = \sum_{J \subseteq [n]} \prod_{j \in J} x_j \prod_{j \notin J} iy_j$~~

Now: $\prod_{j<n} z_j$ for a sequence $\langle z_j : j < n \rangle$ of $z_j \in \mathbf{Q}(i)$

- $\exp\left(\sum_{j<n} \log z_j\right)$ does not really make sense ...
- $z_j \in \mathbf{Z}[i]$: round appx. of $\exp(\sum_{j<n}$ appx. of $\log z_j)$
- $z_j \in \mathbf{Q}(i)$: numerators and denominators separately
- $\prod_{j<0} z_j = 1$, $\prod_{j<n+1} z_j = z_n \prod_{j<n} z_j$

Elementary analytic functions

Using exp and log, define other elementary analytic functions:

- trigonometric: sin, cos, tan, cot, sec, csc
- inverse trig.: arcsin, arccos, arctan, arccot, arcsec, arccsc
- hyperbolic: sinh, cosh, tanh, coth, sech, csch
- inverse hyp.: arsinh, arcosh, artanh, arcoth, arsech, arcsch

They have the expected properties, such as

$$\sin^2 z + \cos^2 z = 1$$
$$\sin(z + w) = \sin z \cos w + \cos z \sin w$$

Example: tangent and arctangent

Define tan: $\mathbf{C} \setminus \pi\left(\frac{1}{2} + \mathbf{Z}\right) \to \mathbf{C}$ by

$$\tan z = \begin{cases} i\dfrac{e^{-iz} - e^{iz}}{e^{-iz} + e^{iz}} & y \in \mathbf{R_L} \\ i \operatorname{sgn} y & y \notin \mathbf{R_L} \end{cases} = \begin{cases} i\dfrac{1 - e^{2iz}}{1 + e^{2iz}} & y \geq 0 \\ i\dfrac{e^{-2iz} - 1}{e^{-2iz} + 1} & y \leq 0 \end{cases}$$

$(z = x + iy)$

and arctan: $\mathbf{C} \setminus \{\pm i\} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] + i\mathbf{R_L}$ by

$$\arctan z = \frac{1}{2i} \log\left(\frac{1 + iz}{1 - iz}\right)$$

Tangent and arctangent (cont'd)

Basic properties:

- arctan continuous outside $\pm i[1, +\infty)$
- $\tan \arctan z = z$ for $z \in \mathbf{C} \smallsetminus \{\pm i\}$
- $\arctan \tan z = z$ for $z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] + i\mathbf{R}_{\mathbf{L}}$, $z \neq \frac{\pi}{2}$
- tan is π-periodic
- $\{w : \tan w = z\} = \arctan z + \pi\mathbf{Z}$ for $z \neq \pm i$
- tan maps $\mathbf{R} \smallsetminus \pi\left(\frac{1}{2} + \mathbf{Z}\right)$ onto $\mathbf{R}$
- arctan maps $\mathbf{R}$ onto $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, increasing
- $\arctan z$ has additive and multiplicative $\mathbf{TC}^0$ approximations for $z \in \mathbf{Q}(i) \smallsetminus \{\pm i\}$

$\mathbf{TC}^0$ approximation of tan more involved:
bound away from singularities!

$\mathbf{TC}^0$ approximation of tangent

$\tan z$ has $\mathbf{TC}^0$ approximations as follows:

- additive for $z \in \mathbf{Q}(i) \setminus \pi\left(\frac{1}{2} + \mathbf{Z}\right)$, parametrized by $r \in \mathbf{L}$ s.t. $z \notin \mathbf{Q}$ or $\operatorname{dist}\left(z, \pi(\frac{1}{2} + \mathbf{Z})\right) \geq 2^{-r}$
- multiplicative for $z \in \mathbf{Q}(i) \setminus \frac{\pi}{2}\mathbf{Z}_{\neq 0}$, parametrized by $r \in \mathbf{L}$ s.t. $z \notin \mathbf{Q}$ or $\operatorname{dist}\left(z, \frac{\pi}{2}\mathbf{Z}_{\neq 0}\right) \geq 2^{-r}$

In the standard model it's much cleaner:

- π irrational, finite irrationality measure $\implies$ $\mathbf{TC}^0$-computable lower bound on $\operatorname{dist}\left(z, \frac{\pi}{2}\mathbb{Z}_{\neq 0}\right)$
- additive and multiplicative approximation for $z \in \mathbb{Q}(i)$

Question: Can VTC^0 prove "π is irrational"?

$\implies$ $\mathbf{TC}^0$ approximation on $\mathbf{Q}(i)$ w/o parameter

Irrationality measure

$x \in \mathbb{R} \setminus \mathbb{Q}$: $\mu(x) = \sup \left\{ \mu : \exists^\infty \langle p, q \rangle \in \mathbb{Z}^2 \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu} \right\}$

- $\mu(x) \geq 2$
- $\mu(x) = 2$ for x algebraic and for almost all $x \in \mathbb{R}$
- $\mu(\pi) \leq 42$ [Mah'53], $\mu(\pi) \leq 7.1032\ldots$ [ZZ'20]
- conjecture: $\mu(\pi) = 2$

$\mu > \mu(\pi) \implies$

$$\operatorname{dist}\left(\frac{p}{q}, \pi\mathbb{Z}\right) \geq \frac{N}{(qN)^\mu} \approx \frac{1}{q}\left(\frac{\pi}{p}\right)^{\mu-1}, \quad N = \left\lfloor \frac{p}{q\pi} \right\rceil$$

for sufficiently large q

Exponential integer parts

Motivation

Let $\langle R, +, \cdot, < \rangle$ be an ordered field

Integer part (IP): subring $D \subseteq R$ s.t.

- D discrete (1 is a least positive element)
- $\forall x \in R \, \exists u \in D \, |x - u| < 1$

Real-closed field (RCF):

- odd-degree $f \in R[x]$ have roots, $\forall x > 0 \, \exists \sqrt{x}$
- equivalently: $R \equiv \mathbb{R}$

Theorem [Shep'64]:

$$\mathfrak{M} \vDash \text{IOpen} \iff \mathfrak{M} \text{ is an IP of a RCF}$$

Exponential integer parts

Exponential field: $\langle R, \exp\rangle$ s.t.

- R ordered field
- $\exp\colon \langle R, +, <\rangle \simeq \langle R_{>0}, \cdot, <\rangle$

Following [Res'93]:

- exponential integer part (EIP):
  IP $D \subseteq R$ s.t. $D_{>0}$ closed under exp
- real-closed exponential field (RCEF):
  exponential field s.t. R RCF, $\exp(1) = 2$, $\exp(x) > x$

NB: $\exp \restriction D_{>0}$ may be different from the usual 2^n

Question:

- What models are EIP of RCEF? Do they satisfy some nontrivial consequences of totality of exponentiation?

Our results

Theorem: Every countable $\mathfrak{M} \vDash \mathrm{VTC}^0$ is an EIP of a RCEF

- uncountable $\mathfrak{M} \vDash \mathrm{VTC}^0$ has an elementary extension to an EIP of a RCEF
- FO consequences of being an EIP of a RCEF are nowhere near $\mathrm{I}\Delta_0 + \mathrm{EXP}$

Starting point:

- $\mathfrak{M} \vDash \mathrm{VTC}^0 \implies$ IP of RCF $\mathbf{R}^{\mathfrak{M}}$
- $2^x \colon \langle \mathbf{R}_{\mathbf{L}}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{R}_{>0}^{\mathfrak{M}}, \cdot, < \rangle$ not quite right
- need $f \colon \langle \mathbf{R}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{R}_{\mathbf{L}}^{\mathfrak{M}}, +, < \rangle$ s.t. $f[\mathbf{Z}^{\mathfrak{M}}] \subseteq \mathbf{Z}_{\mathbf{L}}^{\mathfrak{M}}$

Outline of strategy

Theorem [BS'76]: $\mathfrak{A} \equiv \mathfrak{B}$ countable, $\langle \mathfrak{A}, \mathfrak{B} \rangle$ recursively saturated $\implies \mathfrak{A} \simeq \mathfrak{B}$

- $\mathbf{R}^{\mathfrak{M}}$ is uncountable $\implies$ work with $\mathbf{Q}^{\mathfrak{M}}$ instead
- axiomatize the theory of $\langle \mathbf{Q}, \mathbf{Z}, \mathbf{Q}_{\mathsf{L}}, +, < \rangle$, quantifier elimination
- $\mathfrak{M} \vDash \mathsf{VTC}^0 \implies \langle \mathbf{Q}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, \mathbf{Q}_{\mathsf{L}}^{\mathfrak{M}}, +, < \rangle$ rec. sat.
- $\mathfrak{M}$ countable $\implies \langle \mathbf{Q}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{Q}_{\mathsf{L}}^{\mathfrak{M}}, \mathbf{Z}_{\mathsf{L}}^{\mathfrak{M}}, +, < \rangle$, continuous extension $\langle \mathbf{R}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{R}_{\mathsf{L}}^{\mathfrak{M}}, \mathbf{Z}_{\mathsf{L}}^{\mathfrak{M}}, +, < \rangle$
- alternatively: $\langle \mathbf{Z}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{Z}_{\mathsf{L}}^{\mathfrak{M}}, +, < \rangle$ combines with $\mathrm{id} \colon [0,1] \to [0,1]$ to $\langle \mathbf{R}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{R}_{\mathsf{L}}^{\mathfrak{M}}, \mathbf{Z}_{\mathsf{L}}^{\mathfrak{M}}, +, < \rangle$
- problem: growth axiom $\exp(x) > x$

The theory of three groups

$\mathcal{L}_{3G} = \langle Z, L, +, 0, 1, < \rangle$

- Z and L unary predicates, treat as sets
- denote the whole universe as Q

3G: $\mathcal{L}_{3G}$-theory with axioms

- $\langle Q, +, 0, < \rangle$ is a divisible totally ordered abelian group
- Z is an integer part of Q with least element 1
- L is a convex subgroup of Q containing 1

NB: implies Z is a $\mathbb{Z}$-group

Example: $\langle \mathbf{Q}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, \mathbf{Q}^{\mathfrak{M}}_{\mathbf{L}}, +, 0, 1, < \rangle$, $\langle \mathbf{R}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, \mathbf{R}^{\mathfrak{M}}_{\mathbf{L}}, +, 0, 1, < \rangle$ are models of 3G for any $\mathfrak{M} \vDash \mathrm{VTC}^0$

Quantifier elimination

Theorem:
Every formula is in 3G equivalent to a Boolean combination of

$$\sum_i n_i \lfloor x_i \rfloor \geq n$$
$$\sum_i n_i \{x_i\} \geq n$$
$$\lfloor x_i \rfloor \equiv k \pmod{m}$$
$$\sum_i n_i x_i \in L$$
$$Q = L$$

$(n_i, n, k, m \in \mathbb{Z},\ 0 \leq k < m)$

Corollary: $3G + Q = L$ and $3G + Q \neq L$ are complete

Recursive saturation of models of 3G

Goal: 3G reducts of nonstandard models of VTC^0 rec. sat.

- overspill + $\mathbf{TC}^0$ truth predicate for $\mathcal{L}_{3G}$ formulas
- fails miserably for $\sum_i n_i x_i \in L$:
  $\mathbf{L}^{\mathfrak{M}}$ not definable in $\mathfrak{M} \vDash \mathrm{VTC}^0$ by any bounded formula

$\Longrightarrow$ need to separate the role of L from the rest

Theorem: $\langle Q, Z, L, +, 0, 1, < \rangle \vDash$ 3G is rec. sat. $\iff$

- $\langle Q, Z, +, 0, 1, < \rangle$ is rec. sat.
- there is no $a \in Q_{>0}$ s.t. $\mathbb{N}a$ is cofinal in L
  or $\mathbb{N}^{-1}a$ is cofinal above L

  (i.e., $\{\frac{1}{n}a : n \in \mathbb{N}\}$ downwards cofinal in $Q_{>0} \smallsetminus L_{>0}$)

3G **reducts of models of** VTC^0

Theorem: $\mathfrak{M} \vDash \mathrm{VTC}^0$ nonstandard $\implies$
$\langle \mathbf{Q}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, \mathbf{Q}_{\mathbf{L}}^{\mathfrak{M}}, +, 0, 1, < \rangle$ recursively saturated

Proof sketch:

- it's enough to prove it for $\langle \mathbf{Q}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, +, 0, 1, < \rangle$
- fixing $\vec{a} \in \mathbf{Q}^{\mathfrak{M}}$, we consider sets Γ of disjunctions of

$$n\{x\} > \sum_i n_i a_i \qquad\qquad n\lfloor x \rfloor \geq \sum_i n_i a_i$$

$$\lfloor x \rfloor \equiv k \pmod{m} \qquad\qquad a_i \equiv k \pmod{m}$$

  (unary m, the same for all)
- we can determine whether x satisfies such a Γ by a $\mathbf{TC}^0$ predicate $T(\Gamma, x, \vec{a})$

Overspill argument

- if Γ as above is satisfiable, we can compute a satisfying x by a $\mathbf{TC}^0$ function $S(\Gamma, \vec{a})$
  - sort all the $\frac{1}{n}\sum_i n_i a_i$, try all choices of $\lfloor x \rfloor \bmod m$, ...

$\{\varphi_t(x, \vec{a}) : t \in \mathbb{N}\}$ $\mathbf{TC}^0$-computable sequence (t unary) of disjunctions as above, finitely satisfiable:

- $\forall t \le n\, T\big(\varphi_t, S(\{\varphi_t : t \le n\}, \vec{a}), \vec{a}\big)$ is a $\mathbf{TC}^0$ formula
- it holds for all standard n

  $\Longrightarrow$ it holds for some nonstandard n

  $\Longrightarrow$ $x := S(\{\varphi_t : t \le n\}, \vec{a})$ satisfies $\{\varphi_t(x, \vec{a}) : t \in \mathbb{N}\}$

Isomorphisms

Corollary: $\mathfrak{M} \vDash \mathrm{VTC}^0$ countable
$\implies \langle \mathbf{Q}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{Q}_{\mathbf{L}}^{\mathfrak{M}}, \mathbf{Z}_{\mathbf{L}}^{\mathfrak{M}}, +, < \rangle$
$\implies \langle \mathbf{R}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{R}_{\mathbf{L}}^{\mathfrak{M}}, \mathbf{Z}_{\mathbf{L}}^{\mathfrak{M}}, +, < \rangle$
$\implies$ $\mathbf{R}^{\mathfrak{M}}$ expands to an exponential field with EIP $\mathfrak{M}$

Not yet quite RCEF: missing $\exp(x) > x$

Need $f \colon \langle \mathbf{R}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, +, < \rangle \simeq \langle \mathbf{R}_{\mathbf{L}}^{\mathfrak{M}}, \mathbf{Z}_{\mathbf{L}}^{\mathfrak{M}}, +, < \rangle$ s.t. $2^{f(x)} > x$

Does not follow from recursive saturation as such, but we can adapt the back-and-forth proof of the [BS'76] isomorphism theorem

Summary

Theorem: $\mathfrak{M} \vDash \mathrm{VTC}^0$ nonstandard $\Longrightarrow$ $\langle \mathbf{Q}^{\mathfrak{M}}, \mathbf{Z}^{\mathfrak{M}}, \mathbf{Q}_{\mathbf{L}}^{\mathfrak{M}}, +, 0, 1, < \rangle$ recursively saturated

Theorem: Every countable $\mathfrak{M} \vDash \mathrm{VTC}^0$ is an EIP of a RCEF

Corollary: Every $\mathfrak{M} \vDash \mathrm{VTC}^0$ has an elementary extension to an EIP of a RCEF

Question: What exactly happens for uncountable $\mathfrak{M} \vDash \mathrm{VTC}^0$?

Question: Does every $\mathfrak{M} \vDash \mathrm{IOpen}$ have an elementary extension to an EIP of a RCEF?

Question: Can VTC^0 prove "π is irrational"?

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