Boolean algebras

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We are aiming at decidability and quantifier elimination for the theory of all Boolean algebras, originally proved by A. Tarski. There will be a lot of definitions, but the actual argument is rather simple.

Definition 1 Let BA denote the theory of *Boolean algebras*: structures $\mathbf{B} = \langle B, 0, 1, \wedge, \vee, \neg \rangle$ satisfying the identities[1]

$$\begin{aligned} x \wedge y &= y \wedge x, & x \vee y &= y \vee x, \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z), & x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z), \\ x \wedge 1 &= x, & x \vee 0 &= x, \\ x \wedge \neg x &= 0, & x \vee \neg x &= 1. \end{aligned}$$

Any Boolean algebra carries a partial order $x \leq y \iff x \wedge y = x \iff x \vee y = y$; conversely, all the operations $0, 1, \wedge, \vee, \neg$ are definable in terms of $\leq$, hence we may also think of BA as a theory in the relational language $\{\leq\}$ when this turn out more convenient. We define the difference $x - y = x \wedge \neg y$ and symmetric difference $x \mathbin{\triangle} y = (x - y) \vee (y - x)$, and we write x^e for $e \in \{0, 1\}$ by $x^1 = x$, $x^0 = \neg x$. More generally, for a t-tuple of variables or terms $\vec{x} = \langle x_i : i < t \rangle$ and $\vec{e} \in \{0,1\}^t$, we put[2] $\vec{x}^e = \bigwedge_{i<t} x_i^{e_i}$.

An *atom* in $\mathbf{B}$ is an element $a > 0$ such that $0 < x < a$ for no $x \in B$, i.e., the interval $[0, a]$ has exactly two elements. The set of atoms of $\mathbf{B}$ is denoted $\mathcal{A}(\mathbf{B})$. An algebra is called *atomless* if it has no atoms, and *atomic* if for every $x > 0$, there is an atom $0 < a \leq x$; equivalently, every element $x \in B$ is the join (= supremum) of all atoms below it: $x = \bigvee\bigl(\mathcal{A}(\mathbf{B}) \cap [0, x]\bigr)$. (NB: This includes the claim that the join exists; in general, joins of infinite subsets of B need not exist.) E.g., the powerset algebras $\langle \mathcal{P}(X), \varnothing, X, \cap, \cup, {}^c \rangle$ are atomic. The trivial algebra $\mathbf{1}$ is the only Boolean algebra that is both atomic and atomless.

Recall that we proved earlier:

Theorem 2 *The theory of nontrivial atomless Boolean algebras is complete, ω-categorical, and it has quantifier elimination. Its unique countable model is the Fraïssé limit of the class of nontrivial finite Boolean algebras, and it is also the free Boolean algebra on countably many generators $\mathbf{F}_\omega$.* □

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[1] This axiomatization is due to Huntington. It implies the idempotence, associativity, and absorption laws that are often included among the axioms.

[2] We should write it as $\vec{x}^{\vec{e}}$, but I'll use plain $\vec{x}^e$ to make the notation less heavy.

The algebras at the opposite end of the spectrum are also quite manageable. We formulate this case separately as a warm-up for the general results on BA, and because it will be useful later when we discuss the Feferman–Vaught theorem. I believe this result is due to Skolem:

Theorem 3 *Every formula $\varphi(\vec{x})$ is over the theory of atomic Boolean algebras equivalent to a propositional*[3] *combination of formulas expressing $|\vec{x}^e| \geq s$ for $\vec{e} \in \{0,1\}^t$ and $s \in \mathbb{N}_{>0}$, where $|x|$ denotes the number of atoms below x. In particular, two atomic Boolean algebras are elementarily equivalent iff they have the same number of atoms or they are both infinite.*

Proof: We claim that

$$\mathbf{A}, \vec{a}\ E_k\ \mathbf{B}, \vec{b} \iff \forall \vec{e} \in \{0,1\}^t\ \min\{2^k, |\vec{a}^e|\} = \min\{2^k, |\vec{b}^e|\}$$

defines a graded back-and-forth system on the class of atomic Boolean algebras, from which the result follows. Obviously, each E_k is an equivalence relation with finitely many equivalence classes.

$\mathbf{A}, \vec{a}\ E_0\ \mathbf{B}, \vec{b}$ just means that $\vec{a}^e = 0 \iff \vec{b}^e$ for all $\vec{e} \in \{0,1\}^t$. That this implies $\vec{a} \equiv_0 \vec{b}$ is easiest to check if we formulate the theory in the language $\{\leq\}$, in which case we just need to observe that $a_i \leq a_j \iff a_i - a_j = 0$.

Let $\mathbf{A}, \vec{a}\ E_{k+1}\ \mathbf{B}, \vec{b}$ and $c \in A$; we need to find $d \in B$ such that $\mathbf{A}, \vec{a}, c\ E_k\ \mathbf{B}, \vec{b}, d$. Assume first that $t = 0$ (i.e., there are no $\vec{a}$ and $\vec{b}$), so that we only need $d \in B$ such that $\min\{2^k, |d^\varepsilon|\} = \min\{2^k, |c^\varepsilon|\}$ for $\varepsilon \in \{0,1\}$. We distinguish cases depending on how $|c^\varepsilon|$ compare to 2^k.

Case 1: $|c^0|, |c^1| < 2^k$. Then $|1^{\mathbf{A}}| = |c^0| + |c^1| < 2^{k+1}$, hence $\mathbf{A}\ E_{k+1}\ \mathbf{B}$ implies $|1^{\mathbf{B}}| = |1^{\mathbf{A}}|$, i.e., $\mathbf{A}$ and $\mathbf{B}$ have the same number of atoms. Let d be the join of $|c^1|$ atoms, thus $\neg d$ has the remaining $|c^0|$ atoms below it, and we are done.

Case 2: $|c^1| < 2^k \leq |c^0|$. Then $\mathbf{A}\ E_{k+1}\ \mathbf{B}$ gives that $\mathbf{B}$ has at least $2^k + |c^1| \leq \min\{2^{k+1}, |1^{\mathbf{A}}|\}$ atoms, hence we can choose d as a join of $|c^1|$ atoms, and then $\neg d$ will be above $\geq 2^k$ atoms. Thus, $c\ E_k\ d$ as required.

Case 3: $|c^0| < 2^k \leq |c^1|$ is symmetric.

Case 4: $2^k \leq |c^0|, |c^1|$. Then $\mathbf{B}$, like $\mathbf{A}$, has $\geq 2^{k+1}$ atoms, hence we can choose d as a join of 2^k atoms, and then $\neg d$ will be above $\geq 2^k$ atoms as well.

In the general situation, we apply this argument separately for each $\vec{e} \in \{0,1\}^t$ (with $\vec{a}^e$ and $\vec{b}^e$ in place of $1^{\mathbf{A}}$ and $1^{\mathbf{B}}$, resp.) to find elements $\{d_{\vec{e},\varepsilon} : \vec{e} \in \{0,1\}^t, \varepsilon \in \{0,1\}\}$ such that $d_{\vec{e},1} \wedge d_{\vec{e},0} = 0$, $d_{\vec{e},1} \vee d_{\vec{e},0} = \vec{b}^e$, and $\min\{2^k, |d_{\vec{e},\varepsilon}|\} = \min\{2^k, |\vec{a}^e \wedge c^\varepsilon|\}$. Since $\{\vec{b}^e : \vec{e} \in \{0,1\}^t\}$ is a disjoint partition of 1, the element $d = \bigvee_{\vec{e} \in \{0,1\}^t} d_{\vec{e},1}$ then satisfies $\vec{b}^e \wedge d^\varepsilon = d_{\vec{e},\varepsilon}$ for each $\vec{e}$ and ε, hence $\vec{a}, c\ E_k\ \vec{b}, d$ as required. □

We now turn to the case of general Boolean algebras. One idea we can extract from the proof of Theorem 3 is that the first-order properties of a tuple $\vec{a}$ are determined by sentences true in the parts of the algebra below $\vec{a}^e$ for each $\vec{e} \in \{0,1\}^t$. More formally:

Definition 4 If $\mathbf{B} = \langle B, 0, 1, \wedge, \vee, \neg \rangle$ is a Boolean algebra and $u \in B$, let $\mathbf{B}_u$ denote the Boolean algebra $\langle [0,u], 0, u, \wedge, \vee, u - x \rangle$.

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[3] I've been calling this *Boolean combination* throughout the course, but here I won't for obvious reasons.

Note that $\mathbf{B}_u$ is not quite a subalgebra of $\mathbf{B}$, as it has a different top element and complement. It is, in fact, a *quotient* (homomorphic image) of $\mathbf{B}$ as $x \mapsto x \wedge u$ is a homomorphism $\mathbf{B} \to \mathbf{B}_u$. Crucially, the combined map $x \mapsto \langle x \wedge u, x - u\rangle$ is an isomorphism $\mathbf{B} \simeq \mathbf{B}_u \times \mathbf{B}_{\neg u}$ with inverse homomorphism $\langle y, z\rangle \mapsto y \vee z$, thus an element $u \in B$ determines a splitting of $\mathbf{B}$ as the product of two algebras. Under this splitting, u maps to $\langle 1_u, 0\rangle$ (where we suggestively write $1_u = u$ for the top element of $\mathbf{B}_u$); i.e., $\langle \mathbf{B}, u\rangle \simeq \langle \mathbf{B}_u, 1\rangle \times \langle \mathbf{B}_{\neg u}, 0\rangle$. Using the fact we proved earlier that products preserve elementary equivalence, we see that the set of *formulas* satisfied by u in $\mathbf{B}$ is determined by *sentences* true in the algebras $\mathbf{B}_u$ and $\mathbf{B}_{\neg u}$. The converse also holds as $\mathbf{B}_u$ and $\mathbf{B}_{\neg u}$ are definable in $\mathbf{B}$ with parameter u.

All this immediately generalizes to longer t-tuples of elements:

Lemma 5 *If* $\mathbf{B}$ *is a Boolean algebra and* $\vec{b} \in B^t$*, then*

$$\langle \mathbf{B}, \vec{b}\rangle \simeq \prod_{\vec{e} \in \{0,1\}^t} \langle \mathbf{B}_{\vec{b}^e}, \vec{e}\rangle.$$

Consequently,

$$\mathbf{A}, \vec{a} \equiv \mathbf{B}, \vec{b} \iff \forall \vec{e} \in \{0,1\}^t \, \mathbf{A}_{\vec{a}^e} \equiv \mathbf{B}_{\vec{b}^e}. \qquad \square$$

This allows us to concentrate on classifying Boolean algebras up to elementary equivalence.

Recall that quotients of Boolean algebras in general are determined by ideals:

Definition 6 A subset $I \subseteq B$ of a Boolean algebra $\mathbf{B}$ is an *ideal* if $0 \in I$ and I is closed downwards and under $\vee$. An ideal I defines a congruence $x \sim_I y \iff x \mathbin{\triangle} y \in I$, hence a quotient algebra $\mathbf{B}/I = \mathbf{B}/{\sim_I}$; conversely, if $f \colon \mathbf{B} \to \mathbf{C}$ is a homomorphism, then $f^{-1}[0]$ is an ideal of $\mathbf{B}$, and the image of f as a subalgebra of $\mathbf{C}$ is canonically isomorphic to $\mathbf{B}/I$.

For example, $\mathbf{B}_u \simeq \mathbf{B}/[0, \neg u]$. (In many contexts, it is more natural to define quotients of Boolean algebras using the dual notion of filters rather than ideals: e.g., $\mathbf{B}_u \simeq \mathbf{B}/[u, 1]$. But in what follows, we will naturally encounter ideals, hence this viewpoint is more convenient.)

Our goal is to completely describe the first-order theory of an algebra $\mathbf{B}$ by suitable numerical "invariants". Generalizing from Theorems 2 and 3, some obvious invariants are the number of atoms, and a flag indicating whether the algebra is atomic or whether it also has an atomless part. But this is not enough: the atomic and atomless parts of an algebra do not necessarily exhaust it, hence we will need to iterate this construction after factoring out what we already accounted for.

Definition 7 Let $\mathbf{B}$ be a Boolean algebra. An element $u \in B$ is called *atomic* or *atomless* whenever $\mathbf{B}_u$ is; i.e., u is atomic iff $u = \bigvee\bigl(\mathcal{A}(\mathbf{B}) \cap [0, u]\bigr)$, and u is atomless iff $\mathcal{A}(\mathbf{B}) \cap [0, u] = \varnothing$. We define the ideals

$$\begin{aligned}
\mathcal{I}_{\mathrm{a}}(\mathbf{B}) &= \{u \in B : u \text{ is atomic}\},\\
\mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{B}) &= \{u \in B : u \text{ is atomless}\},\\
\mathcal{I}(\mathbf{B}) &= \{u \vee v : u \in \mathcal{I}_{\mathrm{a}}(\mathbf{B}), v \in \mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{B})\}.
\end{aligned}$$

Clearly, $\mathcal{I}_{\mathrm{a}}(\mathbf{B}) \cap \mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{B}) = \{0\}$. We put $\mathbf{B}' = \mathbf{B}/\mathcal{I}(\mathbf{B})$, and by induction on $n \in \omega$, $\mathbf{B}^{(0)} = \mathbf{B}$, $\mathbf{B}^{(n+1)} = (\mathbf{B}^{(n)})'$ for $n \in \omega$. Notice that each $\mathbf{B}^{(n)}$ can be written as a quotient of the original algebra: $\mathbf{B}^{(n)} \simeq \mathbf{B}/\mathcal{I}_n$ where $\mathcal{I}_0(\mathbf{B}) = \{0\}$, and $\mathcal{I}_{n+1}(\mathbf{B}) = \pi_n^{-1}\bigl[\mathcal{I}(\mathbf{B}/\mathcal{I}_n(\mathbf{B}))\bigr]$ is the lift of $\mathcal{I}(\mathbf{B}/\mathcal{I}_n(\mathbf{B}))$ to $\mathbf{B}$, where π_n denotes the quotient map $\mathbf{B} \to \mathbf{B}/\mathcal{I}_n(\mathbf{B})$.

The *level* $\ell(\mathbf{B})$ of $\mathbf{B}$ is $p < \omega$ if p is minimal such that $\mathbf{B}^{(p+1)} \simeq \mathbf{1}$, and $\ell(\mathbf{B}) = \infty$ if no such p exists. The *invariant* of $\mathbf{B}$, denoted $\mathrm{Inv}(\mathbf{B})$, is ∞ if $\ell(\mathbf{B}) = \infty$; otherwise, $\mathrm{Inv}(\mathbf{B}) = \langle p, q, r\rangle$ where $p = \ell(\mathbf{B}) < \omega$, $q = |\mathcal{A}(\mathbf{B}^{(p)})| \in \omega \cup \{\infty\}$ (i.e., $q = \infty$ whenever $\mathbf{B}^{(p)}$ has infinitely many atoms), and $r \in \{0, 1\}$ indicates whether $\mathbf{B}^{(p)}$ has a nonzero atomless element (i.e., if $\mathbf{B}^{(p)}$ is non-atomic). We have $q + r > 0$ unless $\mathbf{B} \simeq \mathbf{1}$ (in which case $\mathrm{Inv}(\mathbf{B}) = \langle 0, 0, 0\rangle$). Let

$$\mathrm{INV} = \bigl\{\langle p, q, r\rangle \in \omega \times (\omega \cup \{\infty\}) \times \{0, 1\} : q = r = 0 \to p = 0\bigr\} \cup \{\infty\}$$

denote the set of all possible invariants. An invariant $t \in \mathrm{INV}$ is *finite* if $t = \langle p, q, r\rangle$ with $q < \omega$.

Observe that $\mathbf{B}' \simeq \mathbf{1}$ iff $\mathcal{I}(\mathbf{B}) = B$ iff $1 = u \vee v$ with $u \in \mathcal{I}_{\mathrm{a}}(\mathbf{B})$ and $v \in \mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{B})$ (thus $v = \neg u$) iff $u = \bigvee \mathcal{A}(\mathbf{B})$ exists. In particular, $\mathbf{B}' \simeq \mathbf{1}$ whenever $\mathcal{A}(\mathbf{B})$ is finite or $\mathbf{B}$ is atomic. Thus, if $\ell(\mathbf{B}) = p$, then for each $n < p$, $\mathbf{B}^{(n)}$ is a non-atomic algebra with infinitely many atoms, and for each $n > p$, $\mathbf{B}^{(n)} \simeq \mathbf{1}$.

Example 8 $\mathrm{Inv}(\mathcal{P}(X)) = \langle 0, |X|, 0\rangle$ (again, interpreted so that $|X| = \infty$ if X is infinite), $\mathrm{Inv}(\mathbf{F}_\omega) = \langle 0, 0, 1\rangle$, and $\mathrm{Inv}(\mathcal{P}(X) \times \mathbf{F}_\omega) = \langle 0, |X|, 1\rangle$.

We intend to show that $\mathrm{Th}(\mathbf{B})$ is completely described by $\mathrm{Inv}(\mathbf{B})$. We first need to establish that the invariants can be described by first-order sentences, for which we need to show that the ideals $\mathcal{I}_n(\mathbf{B})$ are definable by suitable formulas $\iota_n(x)$, i.e., $\mathcal{I}_n(\mathbf{B}) = \{u \in B : \mathbf{B} \vDash \iota_n(u)\}$ for any $\mathbf{B}$. We take $\iota_0(x) \equiv x = 0$, and for $n = 1$, we can write

$$\begin{aligned}
\mathrm{Atom}(x) &\equiv x \neq 0 \wedge \forall y\,\bigl((x \wedge y) = 0 \vee (x \wedge y) = x\bigr),\\
\mathrm{Atomic}(x) &\equiv \forall y\,\bigl((x \wedge y) \neq 0 \to \exists z\,\bigl(\mathrm{Atom}(z) \wedge (y \wedge z) \neq 0\bigr)\bigr),\\
\mathrm{Atomless}(x) &\equiv \forall y\,\neg\mathrm{Atom}(x \wedge y),\\
\iota_1(x) &\equiv \exists y\,\bigl(\mathrm{Atomic}(x \wedge y) \wedge \mathrm{Atomless}(x \wedge \neg y)\bigr)
\end{aligned}$$

(these are intentionally written in the basic language without $\leq$, even though the latter would be more natural to use). Then we proceed by induction on n: if we already have $\iota_n(x)$, then $\mathbf{B}/\mathcal{I}_n(\mathbf{B})$ is interpretable in $\mathbf{B}$ by an interpretation—that we will also denote ι_n—which leaves all function symbols and the domain absolute, and interprets $x = y$ as $\iota_n(x \mathbin{\triangle} y)$. Then we define $\iota_{n+1}(x)$ as $(\iota_1(x))^{\iota_n}$ (i.e., the definition of $\mathcal{I}$ as interpreted by ι_n). That is, we take $\iota_1(x)$ and replace each atomic subformula $t = s$ with $\iota_n(t \mathbin{\triangle} s)$.

Next, for each $p, q < \omega$, we can define a sentence

$$\alpha_{p,q} \equiv \bigl(|1| \geq q\bigr)^{\iota_p} \equiv \Bigl(\exists x_0, \dots, x_{q-1} \bigwedge_{i<q} \mathrm{Atom}(x_i) \wedge \bigwedge_{i<j<q} x_i \neq x_j\Bigr)^{\iota_p}$$

expressing that $\mathbf{B}^{(p)}$ has at least q atoms, and

$$\begin{aligned}
\beta_p &\equiv \neg\bigl(\mathrm{Atomic}(1)\bigr)^{\iota_p},\\
\gamma_p &\equiv \alpha_{p,1} \vee \beta_p
\end{aligned}$$

expressing that $\mathbf{B}^{(p)}$ is non-atomic, and $\mathbf{B}^{(p)} \not\simeq \mathbf{1}$, respectively.

Then every finite invariant $t = \langle p, q, r\rangle$ is defined by the sentence

$$\tau_t = \alpha_{p,q} \wedge \neg\alpha_{p,q+1} \wedge \beta_p^r$$

(where $\varphi^1 = \varphi$, $\varphi^0 = \neg\varphi$) in the sense that

$$\operatorname{Inv}(\mathbf{B}) = t \iff \mathbf{B} \vDash \tau_t,$$

and every infinite $t \in \mathrm{INV}$ is likewise definable by a theory τ_t: say,

$$\begin{aligned}\tau_{p,\infty,r} &= \{\alpha_{p,q} : q < \omega\} \cup \{\beta_p^r, \neg\gamma_{p+1}\},\\ \tau_\infty &= \{\gamma_n : n < \omega\}\end{aligned}$$

(where $\neg\gamma_{p+1}$ is redundant if $r = 0$). We can summarize this as follows:

Lemma 9 *For every $t \in \mathrm{INV}$, there is a theory τ_t such that*

$$\operatorname{Inv}(\mathbf{B}) = t \iff \mathbf{B} \vDash \tau_t.$$

τ_t consists of propositional combinations of the sentences $\alpha_{p,q}$ and β_p, and if t is finite, then τ_t is a single sentence. In particular,

$$\mathbf{A} \equiv \mathbf{B} \implies \operatorname{Inv}(\mathbf{A}) = \operatorname{Inv}(\mathbf{B}). \qquad \square$$

For the proof of the converse implication, it will be useful to endow INV with a bit of algebraic structure.

Definition 10 We define a relation $\leq \subseteq \mathrm{INV}^2$ and an operation $+\colon \mathrm{INV}^2 \to \mathrm{INV}$ as follows. Let $t, s \in \mathrm{INV}$. If t has strictly smaller level than s, we put $t \leq s$, $s \not\leq t$, and $t + s = s + t = s$. Also $\infty \leq \infty$ and $\infty + \infty = \infty$. If t and s have the same finite level, say $t = \langle p, q, r\rangle$ and $s = \langle p, q', r'\rangle$, we define

$$\begin{aligned}\langle p, q, r\rangle &\leq \langle p, q', r'\rangle \equiv q \leq q' \wedge r \leq r',\\ \langle p, q, r\rangle &+ \langle p, q', r'\rangle = \langle p, q + q', \max\{r \vee r'\}\rangle.\end{aligned}$$

The following can be checked readily from the definition:

Observation 11 *$\leq$ is a partial order on INV with least element $\langle 0, 0, 0\rangle$ and largest element ∞, $+$ is a commutative monoid with neutral element $\langle 0, 0, 0\rangle$ and absorbing element ∞, and we have*

$$t \leq s \iff \exists u \in \mathrm{INV}\; s = t + u. \qquad \square$$

The intention behind $\leq$ and $+$ is that they capture the effect of products on invariants.

Lemma 12 *If $\mathbf{A}$ and $\mathbf{B}$ are Boolean algebras, then $\operatorname{Inv}(\mathbf{A} \times \mathbf{B}) = \operatorname{Inv}(\mathbf{A}) + \operatorname{Inv}(\mathbf{B})$. In particular, $\operatorname{Inv}(\mathbf{B}_u) \leq \operatorname{Inv}(\mathbf{B})$ for any $u \in B$.*

Proof: We check from the definition that $\mathcal{A}(\mathbf{A}\times\mathbf{B}) = \bigl(\mathcal{A}(\mathbf{A})\times\{0\}\bigr)\cup\bigl(\{0\}\times\mathcal{A}(\mathbf{B})\bigr)$, thus $\mathcal{I}_{\mathrm{a}}(\mathbf{A}\times\mathbf{B}) = \mathcal{I}_{\mathrm{a}}(\mathbf{A})\times\mathcal{I}_{\mathrm{a}}(\mathbf{B})$, $\mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{A}\times\mathbf{B}) = \mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{A})\times\mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{B})$, and $\mathcal{I}(\mathbf{A}\times\mathbf{B}) = \mathcal{I}(\mathbf{A})\times\mathcal{I}(\mathbf{B})$, whence $(\mathbf{A}\times\mathbf{B})'\simeq\mathbf{A}'\times\mathbf{B}'$. By induction on n, we get $\mathcal{I}_n(\mathbf{A}\times\mathbf{B}) = \mathcal{I}_n(\mathbf{A})\times\mathcal{I}_n(\mathbf{B})$ and $(\mathbf{A}\times\mathbf{B})^{(n)}\simeq\mathbf{A}^{(n)}\times\mathbf{B}^{(n)}$. This implies $\ell(\mathbf{A}\times\mathbf{B}) = \max\{\ell(\mathbf{A}),\ell(\mathbf{B})\}$.

Thus, $\mathrm{Inv}(\mathbf{A}\times\mathbf{B}) = \infty$ if $\mathrm{Inv}(\mathbf{A})$ or $\mathrm{Inv}(\mathbf{B})$ is ∞. Furthermore, if, say, $\ell(\mathbf{A}) < \ell(\mathbf{B}) = p < \infty$, then $(\mathbf{A}\times\mathbf{B})^{(p)}\simeq\mathbf{B}^{(p)}$, hence $\mathrm{Inv}(\mathbf{A}\times\mathbf{B}) = \mathrm{Inv}(\mathbf{B})$.

Finally, if $\ell(\mathbf{A}) = \ell(\mathbf{B}) = p$, then the expressions above for $\mathcal{A}$ and $\mathcal{I}_{\overline{\mathrm{a}}}$ imply that

$$\bigl|\mathcal{A}\bigl((\mathbf{A}\times\mathbf{B})^{(p)}\bigr)\bigr| = \bigl|\mathcal{A}\bigl(\mathbf{A}^{(p)}\times\mathbf{B}^{(p)}\bigr)\bigr| = \bigl|\mathcal{A}\bigl(\mathbf{A}^{(p)}\bigr)\bigr| + \bigl|\mathcal{A}\bigl(\mathbf{B}^{(p)}\bigr)\bigr|$$

and that $(\mathbf{A}\times\mathbf{B})^{(p)}$ has an atomless element iff one of $\mathbf{A}^{(p)}$ or $\mathbf{B}^{(p)}$ does. Thus, $\mathrm{Inv}(\mathbf{A}\times\mathbf{B}) = \mathrm{Inv}(\mathbf{A}) + \mathrm{Inv}(\mathbf{B})$ as required. □

Crucially, we have the following converse, from which we will infer that equality of invariants defines a weak BFS:

Lemma 13 *If* $\mathrm{Inv}(\mathbf{B}) = t + t'$*, there exists* $\mathbf{C}\succeq\mathbf{B}$ *and* $u\in C$ *such that* $\mathrm{Inv}(\mathbf{C}_u) = t$ *and* $\mathrm{Inv}(\mathbf{C}_{\neg u}) = t'$*.*

Proof: Write $t = \langle p,q,r\rangle$ and $t' = \langle p',q',r'\rangle$. If $t = \infty$, this is understood as $p = \infty$ (and q, r undefined); likewise for t'. W.l.o.g. $p\le p'$, and if $p = p' < \infty$, then $q\le q'$.

Assume first that t is finite. If $p < p'$ or $r = 0$, then t' is uniquely determined by the equation $t + t' = \mathrm{Inv}(\mathbf{B})$, hence in view of Lemma 12, it suffices to find $u\in B$ such that $\mathrm{Inv}(\mathbf{B}_u) = t$. Let $\tilde u\in\mathbf{B}^{(p)}$ be the join of q atoms of $\mathbf{B}^{(p)}$, and—if $r = 1$—a nontrivial atomless element of $\mathbf{B}^{(p)}$; let $u\in\mathbf{B}$ be a preimage of $\tilde u$ under the quotient map $\mathbf{B}\to\mathbf{B}/\mathcal{I}_p(\mathbf{B})$. Then $(\mathbf{B}_u)^{(p)} = (\mathbf{B}^{(p)})_{\tilde u}$ (as in the proof of Lemma 12), which has $q < \infty$ atoms (whence $\mathbf{B}_u$ has level p) and which has a nontrivial nonatomic part iff $r = 1$, thus $\mathrm{Inv}(\mathbf{B}_u) = t$.

If $p = p' = \ell(\mathbf{B})$ and $r = 1$, let $v = \neg\bigvee\mathcal{A}(\mathbf{B}^{(p)})$ be the largest atomless element of $\mathbf{B}^{(p)}$, which exists as $\mathbf{B}^{(p+1)} = \mathbf{1}$, and let v' be either v or some $0 < v' < v$ depending on whether r' is 0 or 1. Let $\tilde u\in\mathbf{B}^{(p)}$ be the join of q atoms of $\mathbf{B}^{(p)}$ and v'; let $u\in\mathbf{B}$ be its preimage. As above, $\mathrm{Inv}(\mathbf{B}_u) = t$. Moreover, $(\mathbf{B}_{\neg u})^{(p)}$ has $|\mathcal{A}(\mathbf{B}^{(p)})| - q = q'$ atoms, and it is non-atomic iff $r' = 1$, which (in view of $q' + r' > 0$) shows $\mathrm{Inv}(\mathbf{B}_{\neg u}) = t'$.

If t is infinite, we consider the theory $T = \mathrm{Th}(\mathbf{B}_B) + \tau_t^{[0,u]} + \tau_{t'}^{[0,\neg u]}$ in a language with constants for elements of B, and a fresh constant u, where $\tau_t^{[0,u]} = \{\varphi^{[0,u]} : \varphi\in\tau_t\}$ denotes the theory τ_t from Lemma 9 to which we apply the interpretation that defines the quotient by the ideal $[0,u]$, and similarly for $\tau_{t'}^{[0,\neg u]}$. A model of T is a $\mathbf{C}\succeq\mathbf{B}$ with an element u such that $\mathbf{C}_u\simeq\mathbf{C}/[0,u]\vDash\tau_t$ and $\mathbf{C}_{\neg u}\vDash\tau_{t'}$, i.e., $\mathrm{Inv}(\mathbf{C}_u) = t$ and $\mathrm{Inv}(\mathbf{C}_{\neg u}) = t'$ as required. Thus, using compactness, it suffices to show that every finite subtheory of T has a model.

If $p < \infty$, then either $p < p'$, or $p = p'$ and $q = q' = \infty$ (as t is assumed infinite); in both cases, $\langle p,m,r\rangle + t' = t + t' = \mathrm{Inv}(\mathbf{B})$ for all $m < \infty$, and we may apply the previous part to $\langle p,m,r\rangle$ and t'. This yields an element $u\in B$ such that $\mathrm{Inv}(\mathbf{B}_u) = \langle p,m,r\rangle$ and $\mathrm{Inv}(\mathbf{B}_{\neg u}) = t'$, thus $\langle\mathbf{B}_B,u\rangle$ satisfies all of $\mathrm{Th}(\mathbf{B}_B) + \tau_{t'}^{[0,\neg u]}$, and a finite subset of $\tau_t^{[0,u]}$ that can be made arbitrarily large as m increases.

Similarly, if $p = p' = \infty$, we can find u such that $\mathrm{Inv}(\mathbf{B}_u) = \langle m,0,1\rangle$ and $\mathrm{Inv}(\mathbf{B}_{\neg u}) = \infty$ for arbitrarily large m. □

Before we get to the main results, we still need one more general fact:

Lemma 14 *For any structures* $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{B}$, *and* $\mathbf{B}'$,

$$\mathbf{A} \preceq \mathbf{A}' \text{ and } \mathbf{B} \preceq \mathbf{B}' \implies \mathbf{A} \times \mathbf{B} \preceq \mathbf{A}' \times \mathbf{B}'.$$

Proof: We proved this earlier for elementary equivalence using Ehrenfeucht–Fraïssé games. This implies the version for elementary embeddings if we expand the language with constants for elements of $A \times B$: in $\mathbf{A}$ and $\mathbf{A}'$, we interpret the constant $\underline{\langle a, b\rangle}$ as a, and in $\mathbf{B}$ and $\mathbf{B}'$, we interpret it as b. Then we get

$$\begin{aligned} \mathbf{A} \preceq \mathbf{A}' \text{ and } \mathbf{B} \preceq \mathbf{B}' &\implies \mathbf{A}_{A\times B} \equiv \mathbf{A}'_{A\times B} \text{ and } \mathbf{B}_{A\times B} \equiv \mathbf{B}'_{A\times B} \\ &\implies (\mathbf{A} \times \mathbf{B})_{A\times B} = \mathbf{A}_{A\times B} \times \mathbf{B}_{A\times B} \equiv \mathbf{A}'_{A\times B} \times \mathbf{B}'_{A\times B} = (\mathbf{A}' \times \mathbf{B}')_{A\times B} \\ &\implies \mathbf{A} \times \mathbf{B} \preceq \mathbf{A}' \times \mathbf{B}'. \end{aligned}$$

□

Theorem 15 *For all Boolean algebras* $\mathbf{A}$ *and* $\mathbf{B}$,

$$\mathbf{A} \equiv \mathbf{B} \iff \operatorname{Inv}(\mathbf{A}) = \operatorname{Inv}(\mathbf{B}).$$

Proof: We have the left-to-right implication from Lemma 9. For the converse, it suffices to prove that the relation $\approx \subseteq \bigcup_t \big(\mathrm{Tup}_t(\mathrm{Mod}(\mathsf{BA})) \times \mathrm{Tup}_t(\mathrm{Mod}(\mathsf{BA}))\big)$ defined by

$$\mathbf{A}, \vec{a} \approx \mathbf{B}, \vec{b} \iff \forall \vec{e} \in \{0,1\}^t \ \operatorname{Inv}(\mathbf{A}_{\vec{a}^e}) = \operatorname{Inv}(\mathbf{B}_{\vec{b}^e})$$

is a weak back-and-forth system.

Clearly, $\approx$ is an equivalence relation, and $\mathbf{A}, \vec{a} \approx \mathbf{B}, \vec{b}$ implies $\vec{a}^e = 0 \iff \vec{b}^e = 0$ for every $\vec{e} \in \{0,1\}^t$, which easily gives $\mathbf{A}, \vec{a} \equiv_0 \mathbf{B}, \vec{b}$.

Assume $\mathbf{A}, \vec{a} \approx \mathbf{B}, \vec{b}$ and $c \in A$; we will find $d \in B$ such that $\mathbf{A}, \vec{a}, c \approx \mathbf{B}, \vec{b}, d$.

If $t = 0$, $\mathbf{A} \simeq \mathbf{A}_c \times \mathbf{A}_{\neg c}$ gives $\operatorname{Inv}(\mathbf{A}_c) + \operatorname{Inv}(\mathbf{A}_{\neg c}) = \operatorname{Inv}(\mathbf{A}) = \operatorname{Inv}(\mathbf{B})$ by Lemma 12, thus there is $\mathbf{D} \succeq \mathbf{B}$ and $d \in \mathbf{D}$ such that $\operatorname{Inv}(\mathbf{D}_d) = \operatorname{Inv}(\mathbf{A}_c)$ and $\operatorname{Inv}(\mathbf{D}_{\neg d}) = \operatorname{Inv}(\mathbf{A}_{\neg c})$ by Lemma 13, and we are done.

In the general case, we apply the same argument to each $\mathbf{B}_{\vec{b}^e}$ separately. That is, for each $\vec{e} \in \{0,1\}^t$, $\mathbf{A}_{\vec{a}^e} \simeq \mathbf{A}_{\vec{a}^e \wedge c} \times \mathbf{A}_{\vec{a}^e - c}$ gives $\operatorname{Inv}(\mathbf{A}_{\vec{a}^e \wedge c}) + \operatorname{Inv}(\mathbf{A}_{\vec{a}^e - c}) = \operatorname{Inv}(\mathbf{A}_{\vec{a}^e}) = \operatorname{Inv}(\mathbf{B}_{\vec{b}^e})$ by Lemma 12, thus using Lemma 13, there are $\mathbf{D}^{\vec{e}} \succeq \mathbf{B}_{\vec{b}^e}$ and $d_{\vec{e}} \in \mathbf{D}^{\vec{e}}$ such that $\operatorname{Inv}(\mathbf{A}_{\vec{a}^e \wedge c}) = \operatorname{Inv}(\mathbf{D}^{\vec{e}}_{d_{\vec{e}}})$ and $\operatorname{Inv}(\mathbf{A}_{\vec{a}^e - c}) = \operatorname{Inv}(\mathbf{D}^{\vec{e}}_{\neg d_{\vec{e}}})$. Putting

$$\langle \mathbf{D}, d\rangle = \prod_{\vec{e} \in \{0,1\}^t} \langle \mathbf{D}^{\vec{e}}, d_{\vec{e}}\rangle$$

and recalling

$$\mathbf{B} = \prod_{\vec{e} \in \{0,1\}^t} \mathbf{B}_{\vec{b}^e},$$

we have $\mathbf{B} \preceq \mathbf{D}$ by Lemma 14. Moreover, for each $\langle \vec{e}, \varepsilon\rangle \in \{0,1\}^{t+1}$ we have $\mathbf{D}_{\vec{b}^e \wedge d^\varepsilon} \simeq \mathbf{D}^{\vec{e}}_{d^{\varepsilon}_{\vec{e}}}$, hence $\operatorname{Inv}(\mathbf{D}_{\vec{b}^e \wedge d^\varepsilon}) = \operatorname{Inv}(\mathbf{A}_{\vec{a}^e \wedge c^\varepsilon})$. Thus, $\mathbf{A}, \vec{a}, c \approx \mathbf{B}, \vec{b}, d$ as required. □

Definition 16 For any $p, q < \omega$, let $\tilde{\alpha}_{p,q}(x)$ denote the formula $\alpha_{p,q}^{[0,x]}$ (the quotient-by-$[0,x]$ interpretation applied to the sentence $\alpha_{p,q}$), and likewise, $\tilde{\beta}_p(x) \equiv \beta_p^{[0,x]}$.

Theorem 17 *Every formula $\varphi(x_0, \dots, x_{t-1})$ is over* BA *equivalent to a propositional combination of formulas of the form $\tilde{\alpha}_{p,q}(\vec{x}^e)$ and $\tilde{\beta}_p(\vec{x}^e)$ for some $p, q < \omega$ and $\vec{e} \in \{0,1\}^t$, expressing that* $\mathrm{Inv}(\mathbf{B}_{\vec{x}^e}) \geq \langle p, q, 0\rangle$ *and* $\mathrm{Inv}(\mathbf{B}_{\vec{x}^e}) \geq \langle p, 0, 1\rangle$*, respectively.*

Proof: If $\mathbf{A} \vDash \theta(\vec{a}) \iff \mathbf{B} \vDash \theta(\vec{b})$ for all formulas θ as in the statement, then $\mathbf{A}, \vec{a} \approx \mathbf{B}, \vec{b}$. This gives the result by general properties of weak (G)BFS that we established earlier. ☐

Theorem 18 *The theory* BA *of Boolean algebras is decidable.*

Proof: Theorem 17 and the fact that BA is recursively (even finitely) axiomatized implies that given a sentence φ, we can compute an equivalent sentence ψ which is a propositional combination of sentences $\{\alpha_{p,q}, \beta_p : \langle p, q\rangle \in I\}$ for some finite $I \subseteq \mathbb{N}^2$. For an algebra $\mathbf{B}$ with $\mathrm{Inv}(\mathbf{B}) = t$, we can determine if $\mathbf{B} \vDash \psi$ by checking for which $\langle p, q\rangle \in I$ is $t \geq \langle p, q, 0\rangle$ resp. $t \geq \langle p, 0, 1\rangle$, which gives us the truth value of $\alpha_{p,q}$ and β_p, and evaluating the propositional formula on the resulting assignment. Let us denote this test $t \vDash \psi$ for short. Putting $\Theta = \{\langle 0,0,0\rangle\} \cup \{\langle p,q,0\rangle, \langle p,0,1\rangle, \langle p,q,1\rangle : \langle p,q\rangle \in I\}$, we see that for any $t \in \mathrm{INV}$, there is $t' \leq t$, $t' \in \Theta$ such that $t' \vDash \psi \iff t \vDash \psi$.

If all $t \in \mathrm{INV}$ are realizable as $\mathrm{Inv}(\mathbf{B})$ for some $\mathbf{B}$, then we have $\mathsf{BA} \vdash \varphi \iff \forall t \in \Theta\, t \vDash \psi$, which we can check by an algorithm. But so far we haven't even seen any algebras of level ≥ 1. Fortunately, we do not actually need this.

Put $\mathrm{INV}' = \{\mathrm{Inv}(\mathbf{B}) : \mathbf{B} \vDash \mathsf{BA}\}$. Then $\mathrm{INV}' \subseteq \mathrm{INV}$ is closed downwards and under $+$ by Lemmas 12 and 13, and in fact, it has a largest element $t_{\max}$ (either by considering the infinite product of representatives of *all* possible invariants, which majorizes all by Lemma 12, or by showing closure under limits of chains using compactness). Since this forces $t_{\max} = t_{\max} + t_{\max}$, $t_{\max}$ is either infinite (∞, $\langle p, \infty, r\rangle$) or $\langle p, 0, 1\rangle$. Anyway, we have $\mathrm{INV}' = \{t \in \mathrm{INV} : t \leq t_{\max}\}$, which is a decidable set no matter what is the actual value of $t_{\max}$. That is, we have $\mathsf{BA} \vdash \varphi$ iff $t \vDash \psi$ for all $t \in \Theta$ such that $t \leq t_{\max}$. ☐

As a matter of fact, $t_{\max} = \infty$, i.e., there exist Boolean algebras with arbitrary invariants $t \in \mathrm{INV}$, but it takes a bit of work to show this. The proof below relies on the *Stone duality* between Boolean algebras and *Boolean spaces*, which are compact Hausdorff totally disconnected topological spaces, we will thus assume familiarity with topology from this point on.

Recall that given a Boolean space X, the corresponding Boolean algebra is the algebra of *clopen sets* $\langle \mathrm{CO}(X), \varnothing, X, \cap, \cup, {}^c\rangle \subseteq \mathcal{P}(X)$; given an algebra $\mathbf{B}$, its *Stone space* $\mathrm{St}(\mathbf{B})$ is the set of maximal ideals (or dually, ultrafilters) in $\mathbf{B}$ topologized such that the sets $\{I \in \mathrm{St}(\mathbf{B}) : u \in I\}$ for $u \in \mathbf{B}$ form a basis of (cl)open sets. Stone's theorem ensures that these constructions are mutually inverse (up to isomorphism/homeomorphism), thus in particular, every Boolean algebra is of the form $\mathrm{CO}(X)$ for some Boolean space X. For example, the Stone space of $\mathbf{F}_\omega$ is homeomorphic to the Cantor space $\{0,1\}^\omega$. The construction also works on morphisms: a homomorphism $f \colon \mathbf{A} \to \mathbf{B}$ yields a continuous map $\mathrm{St}(\mathbf{B}) \to \mathrm{St}(\mathbf{A})$, $I \mapsto f^{-1}[I]$, and conversely, a continuous map $f \colon X \to Y$ yields a homomorphism $\mathrm{CO}(Y) \to \mathrm{CO}(X)$, $U \mapsto f^{-1}[U]$. Thus, we have a pair of contravariant functors (in fact, a dual equivalence) between the categories of Boolean algebras and Boolean spaces. In particular, quotients and/or ideals of Boolean algebras correspond to closed subspaces of Boolean spaces.

The following statement implies by induction that there are Boolean algebras of arbitrarily large finite levels:

Theorem 19 *For every Boolean algebra* $\mathbf{B}$*, there exists an algebra* $\mathbf{C}$ *such that* $\mathbf{C}' = \mathbf{B}$*, thus* $\ell(\mathbf{C}) = \ell(\mathbf{B}) + 1$*.*

Proof: First, there exists an atomic Boolean algebra $\mathbf{A}$ and an onto homomorphism $h\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ that maps all atoms to 0: by Stone's theorem, we can embed $\mathbf{B} \subseteq \mathcal{P}(X)$ for some X; we may assume that all nonempty sets in $\mathbf{B}$ are infinite by passing to $\mathcal{P}(X \times \omega)$ and blowing up each $U \in \mathbf{B}$ to $U \times \omega$ if necessary. Put $\mathbf{A} = \{U \mathbin{\triangle} F : U \in \mathbf{B}, F \subseteq X \text{ finite}\}$, which includes all atoms of $\mathcal{P}(X)$, and let $h\colon \mathbf{A} \to \mathbf{B}$ map each $V \in \mathbf{A}$ to the unique $U \in \mathbf{B}$ such that $U \mathbin{\triangle} V$ is finite.

Second, there is an atomless algebra $\mathbf{F}$ and an onto morphism $f\colon \mathbf{F} \twoheadrightarrow \mathbf{B}$ such that for each nonzero $u \in \mathbf{F}$, there is $0 < v < u$ such that $f(v) = 0$: if $\mathbf{B} = \mathrm{CO}(X)$, let $\mathbf{F} = \mathrm{CO}(X \times \{0,1\}^\omega)$ and $f\colon \mathbf{F} \to \mathbf{B}$, $f(u) = \nu^{-1}[u]$ be the map dual to the inclusion $\nu\colon X \to X \times \{0,1\}^\omega$, $\nu(x) = \langle x, 0^\omega\rangle$. (Stated purely algebraically, $\mathbf{F}$ is the free product $\mathbf{B} \otimes \mathbf{F}_\omega$.)

Let $\mathbf{C} = \{\langle u, v\rangle \in \mathbf{A} \times \mathbf{F} : h(u) = f(v)\}$ (i.e., the pullback of f and h). If $a \in \mathcal{A}(\mathbf{A})$, then $\langle a, 0\rangle \in \mathcal{A}(\mathbf{C})$; conversely, if $\langle u, v\rangle \in \mathbf{B}$ has $u \neq 0$, it is above some $\langle a, 0\rangle$, and if $v \neq 0$, it is strictly above some $\langle 0, w\rangle$ with $w \neq 0$, $f(w) = 0$. Thus, $\mathcal{A}(\mathbf{C}) = \mathcal{A}(\mathbf{A}) \times \{0\}$, $\mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{C}) = \{0\} \times f^{-1}[0]$, and $\mathcal{I}_{\mathrm{a}}(\mathbf{C}) = h^{-1}[0] \times \{0\}$, whence $\mathcal{I}(\mathbf{C}) = h^{-1}[0] \times f^{-1}[0]$ is the kernel of the homomorphism $\mathbf{C} \twoheadrightarrow \mathbf{B}$, $\langle u, v\rangle \mapsto f(u) = h(v)$. Thus, $\mathbf{C}' \simeq \mathbf{B}$. ☐

Corollary 20 *For every* $t \in \mathrm{INV}$*, there exists a Boolean algebra* $\mathbf{B}$ *such that* $\mathrm{Inv}(\mathbf{B}) = t$*.* ☐

Example 21 The class of *complete* Boolean algebras (i.e., such that all subsets have suprema) is not itself first-order definable, but we may consider its induced first-order theory anyway. Now, since every complete Boolean algebra has level 0, this theory is strictly stronger than BA (in fact, it is $\mathsf{BA} + \neg\gamma_1$). Note that every Boolean algebra has a regular completion (i.e., it embeds in a complete Boolean algebra in such a way that the value of all existing suprema and infema is preserved), but as seen here, this embedding is not elementary in general.

Remark 22 One checks easily that in terms of Stone duality, Definition 7 translates as follows: if $\mathbf{B} = \mathrm{CO}(X)$ for a Boolean space X, let $I(X)$ denote the set of isolated points of X. Then $\mathcal{A}(\mathbf{B}) = \bigl\{\{x\} : x \in I(X)\bigr\}$; an $A \in \mathrm{CO}(X)$ is in $\mathcal{I}_{\overline{\mathrm{a}}}(\mathbf{B})$ iff $A \subseteq X \smallsetminus I(X)$ iff $A \subseteq X \smallsetminus \overline{I(X)}$; and $A \in \mathcal{I}_{\mathrm{a}}(\mathbf{B})$ iff $A \subseteq \overline{I(X)}$ iff $A \subseteq \mathrm{int}(\overline{I(X)})$. Thus, $A \in \mathcal{I}(\mathbf{B})$ iff A is disjoint from the closed subspace $\partial(\overline{I(X)})$, and $\mathbf{B}' \simeq \mathrm{CO}(\partial(\overline{I(X)}))$. (Here, we write $\overline{A}$ for the closure of $A \subseteq X$, $\mathrm{int}\,A$ for its interior, an $\partial A = \overline{A} \smallsetminus \mathrm{int}\,A$ for its boundary.)

The proof of Theorem 19 can be done entirely in terms of the dual spaces. For the construction of $\mathbf{A}$, if $\mathbf{B} = \mathrm{CO}(X)$, let $\mathbf{A} = \mathrm{CO}(X \times (\omega + 1))$ with a topology whose basis consists of $\{\langle x, n\rangle\}$ for $x \in X$ and $n \in \omega$, and $(U \times (\omega + 1)) \smallsetminus F$ for $U \subseteq X$ open (or clopen) and F finite. Then we take $\mathbf{C} = \mathrm{CO}(Y)$ where Y is $X \times \{0,1\}^\omega$ glued with $X \times (\omega + 1)$ (with the topology above) so that $X \times \{0^\omega\}$ in the former is identified with $X \times \{\omega\}$ in the latter.

References

[1] Chen Chung Chang and H. Jerome Keisler, *Model theory*, third ed., Studies in Logic and the Foundations of Mathematics vol. 73, North-Holland, 1990, §5.5.