Exercises for Mathematical Logic (4 Jan 2023)

27. All Σ_1-definable sets are semidecidable.

28. (Craig's trick.) Every semidecidable theory is recursively axiomatizable. [Hint: Express $\mathrm{Thm}(T)$ as $\exists y\, P(x, y)$ with P decidable. Given $x = \ulcorner \varphi \urcorner$ and y, devise a sentence equivalent to φ that encodes y.]

29. Show that every decidable consistent theory T has a decidable completion. [Hint: Consider a completion procedure that enumerates sentences φ one by one, and extends the current list of axioms with φ or $\neg\varphi$, whichever maintains consistency with T.]

30. Prove Gödel's diagonal lemma: for every formula $\varphi(x)$, there exists a sentence α such that $Q \vdash \alpha \leftrightarrow \varphi(\overline{\ulcorner \alpha \urcorner})$. [Hint: Using representability of a suitable computable function (see Exer. 26), construct a formula $\psi(x)$ such that $Q \vdash \psi(\overline{\ulcorner \chi \urcorner}) \leftrightarrow \varphi(\overline{\ulcorner \chi(\overline{\ulcorner \chi \urcorner}) \urcorner})$ for all $\chi(x)$.]