Exercises for Mathematical Logic (8 Oct 2024)

7. Every Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ can be represented by a formula of size $O(2^n)$. [Hint: Inductively express a formula in $n+1$ variables as a combination of formulas in n variables.]

8. Prove the propositional soundness theorem: for all $\Gamma \subseteq \mathrm{Prop}_A$ and $\varphi \in \mathrm{Prop}_A$, if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

9. Let $\Gamma, \Delta \subseteq \mathrm{Prop}_A$ and $\varphi, \psi \in \mathrm{Prop}_A$. Show that if $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$.

10. For every $\varphi \in \mathrm{Prop}_A$, its *De Morgan dual* $\varphi^{\mathrm{d}} \in \mathrm{Prop}_A$ is obtained by exchanging $\wedge$ with $\vee$, and $\top$ with $\bot$ inside φ. Formally, we define φ^{d} by induction on the complexity of φ:

$$\begin{aligned} a^{\mathrm{d}} &= a, \quad a \in A, & (\neg\varphi)^{\mathrm{d}} &= \neg(\varphi^{\mathrm{d}}), \\ \top^{\mathrm{d}} &= \bot, & \bot^{\mathrm{d}} &= \top, \\ (\varphi \wedge \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \vee \psi^{\mathrm{d}}), & (\varphi \vee \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \wedge \psi^{\mathrm{d}}). \end{aligned}$$

Show that for all assignments $v\colon A \to \{0,1\}$, $v(\varphi^{\mathrm{d}}) = v_{\neg}(\neg\varphi)$, where $v_{\neg}\colon A \to \{0,1\}$ is the assignment defined by $v_{\neg}(a) = 1 - v(a)$ for each $a \in A$.

11. Let $\varphi, \psi \in \mathrm{Prop}_A$.

(i) $\varphi \equiv \psi$ if and only if $\varphi^{\mathrm{d}} \equiv \psi^{\mathrm{d}}$.

(ii) $\varphi \vDash \psi$ if and only if $\psi^{\mathrm{d}} \vDash \varphi^{\mathrm{d}}$.