Exercises for Mathematical Logic (12 Nov 2024)

19. Using Vaught's test, show the completeness of the theory of a successor: it has a language with one unary function symbol s, and axioms $s(x) = s(y) \to x = y$, $\forall x \, \exists y \, s(y) = x$, and $s^n(x) \neq x$ for each $n \in \mathbb{N}_{>0}$, where s^n denotes the n-fold iteration of s (i.e., $s^0(x)$ is x, and s^{n+1} is $s(s^n(x))$).

20. For each $n \in \mathbb{N}$, let P_n denote the path graph of length n, i.e., the structure $\langle [n], E_n \rangle$, where $[n] = \{0, \dots, n-1\}$ and $E_n = \{\langle i, j \rangle \in [n]^2 : |i - j| = 1\}$. Show that there is no sentence φ such that for all $n \in \mathbb{N}$, $P_n \vDash \varphi$ iff n is odd. [Hint: Adapt the previous exercise.]

21. Fix a field F. The theory of vector spaces over F has a language consisting of the language $\{+, -, 0\}$ of abelian groups and unary functions $a \cdot x$ for each $a \in F$; it has the usual algebraic axioms (axioms of abelian groups, $ab \cdot x = a \cdot (b \cdot x)$, $1 \cdot x = x$, $(a+b) \cdot x = a \cdot x + b \cdot x$, $a \cdot (x+y) = a \cdot x + a \cdot y$). Show that the theory of infinite vector spaces over F (i.e., with additional axioms $\exists x_0 \dots \exists x_n \bigwedge_{i<j} x_i \neq x_j$ for $n \in \mathbb{N}$) is complete and κ-categorical for all infinite $\kappa > |F|$. [Hint: Every vector space has a basis.]

22. An *atom* in a Boolean algebra $\mathbf{A} = \langle A, 0, 1, \wedge, \vee, -, \leq \rangle$ is an element $a \in A$ such that $a > 0$, but $0 < x < a$ for no $x \in A$; $\mathbf{A}$ is *atomless* if $0 \neq 1$ and $\mathbf{A}$ has no atoms. Show that the theory of atomless Boolean algebras is $\aleph_0$-categorical, hence complete.
[Hint: Construct an isomorphism between two countable atomless Boolean algebras $\mathbf{A}$ and $\mathbf{B}$ by a back-and-forth argument, as a union of a sequence of isomorphisms between finite subalgebras. It might help to observe that if $\mathbf{A}_0$ is a finite subalgebra of $\mathbf{A}$, and $\mathbf{A}_1$ is the algebra generated by $A_0 \cup \{b\}$ for some $b \in A$, then each atom of $\mathbf{A}_0$ either remains an atom in $\mathbf{A}_1$, or splits into two atoms.]