Exercises for Mathematical Logic (26 Nov 2024)

23. Show that the functions $+\colon \mathbb{N}^2 \to \mathbb{N}$ and $\cdot\colon \mathbb{N}^2 \to \mathbb{N}$ are computable when the input and output are represented in unary.

24. The same when the input and output are represented in binary.

25. Show that there are computable functions converting natural numbers from one representation to another (unary, ordinary base-k, bijective base-k, considering also different k's).

26. Fix an alphabet Σ.

(i) The following functions are computable: the constant function ε; the functions $s_a\colon \Sigma^* \to \Sigma^*$ for $a \in \Sigma$, defined by $s_a(x) = x\smile a$; the projections $\pi^n_i : (\Sigma^*)^n \to \Sigma^*$, $\pi^n_i(x_0, \dots, x_{n-1}) = x_i$.

(ii) If $f\colon (\Sigma^*)^n \to \Sigma^*$ and $g_i\colon (\Sigma^*)^m \to \Sigma^*$, $i < n$, are computable functions, their composition $h\colon (\Sigma^*)^m \to \Sigma^*$, $h(\vec{x}) = f(g_0(\vec{x}), \dots, g_{n-1}(\vec{x}))$, is computable.

(iii) If $f_\varepsilon\colon (\Sigma^*)^n \to \Sigma^*$ and $f_a\colon (\Sigma^*)^{n+2} \to \Sigma^*$, $a \in \Sigma$, are computable, the function $h\colon (\Sigma^*)^{n+1} \to \Sigma^*$ defined from them by the recursion

$$h(\vec{x}, \varepsilon) = f_\varepsilon(\vec{x}),$$
$$h(\vec{x}, y\smile a) = f_a(\vec{x}, y, h(\vec{x}, y))$$

is computable.

Functions in the smallest class that contains the functions from (i) and that is closed under the operations (ii) and (iii) are called *primitive recursive.* (Usually, the definition of primitive recursive functions is stated for functions $\mathbb{N}^n \to \mathbb{N}$, corresponding to our definition with $|\Sigma| = 1$ and the integers represented in unary. Our more general definition is equivalent up to the bijective base-$|\Sigma|$ numeration.)

27. The set of well bracketed strings over the alphabet $\Sigma = \{(_i,)_i : i < k\}$ is the smallest set of strings such that the empty string ε is well bracketed, and if x and y are well bracketed and $i < k$, then xy and $(_i x)_i$ are well bracketed. E.g., $(_3(_1)_1(_2()_0)_2(_1)_1)_3(_2)_2$ is well bracketed. Show that the set of well bracketed strings is decidable.

28. Let L be a finite first-order language. Show that the following sets and functions are computable:

(i) The set of L-terms.

(ii) The set of L-formulas.

(iii) The set of pairs $\langle \varphi, x\rangle$ where x is a free variable of an L-formula φ.

(iv) The substitution function: given an L-formula φ, a variable x, and an L-term t, compute $\varphi(t/x)$.

(v) The set of triples $\langle \Gamma, \varphi, \pi\rangle$ where π is a proof of an L-formula φ from a finite set of L-formulas Γ.