Exercises for Mathematical Logic (3 Dec 2024)

29. A language $X \subseteq \Sigma^*$ is semidecidable iff it can be represented as $\exists w \in \Sigma'^* \, P(x, w)$ for a finite alphabet Σ' (which we might take to be Σ itself if $|\Sigma| \geq 2$) and a decidable predicate P.
[Hint: Consider a description of an accepting run of a Turing machine, or—if you are already familiar with the section on arithmetic—a Σ_1-formula that defines X in $\mathbb{N}$.]

30. (Craig's trick.) Every semidecidable theory is recursively axiomatizable. [Hint: Express $\mathrm{Thm}(T)$ as $\exists w \, P(\varphi, w)$ with P decidable. Given φ and w, devise a sentence equivalent to φ that encodes w.]

31. Show that every decidable consistent theory T has a decidable complete extension.
[Hint: Consider a completion procedure that enumerates sentences φ one by one, and extends the current list of axioms with φ or $\neg\varphi$, whichever maintains consistency with T.]