

Exercises for Mathematical Logic (10 Dec 2024)

32. Prove $\mathbb{Q} \vdash \forall x (x \leq \bar{n} \vee \bar{n} \leq x)$ for each $n \in \mathbb{N}$.

33. \mathbb{Q} proves $x \cdot y = 0 \rightarrow x = 0 \vee y = 0$, and more generally, $x \cdot y = \bar{n} \rightarrow x = 0 \vee y \leq \bar{n}$ for each $n \in \mathbb{N}$.

34. The standard model \mathbb{N} extends to an L_{PA} -structure \mathbb{N}^∞ with domain $\mathbb{N} \cup \{\infty\}$, $\infty \notin \mathbb{N}$, so that $\mathbb{N}^\infty \models \mathbb{Q}$. Moreover, we are free to choose $(0 \cdot \infty)^{\mathbb{N}^\infty}$ in an arbitrary way (while the rest of the model is uniquely determined by the axioms of \mathbb{Q}). Conclude that \mathbb{Q} does not prove any of the formulas $S(x) \not\leq x$, $x \cdot y = y \cdot x$, or $0 \cdot x \neq 1$.

35. \mathbb{Q} does not prove $x + y = y + x$ or $0 + (x + y) = (0 + x) + y$.
[Hint: Modify the previous exercise to a model with two “infinities”.]