Exercises for Mathematical Logic (7 Jan 2025)

A formula $\varphi(x)$ *represents* a set $X \subseteq \mathbb{N}$ in a theory T if $T \vdash \varphi(\overline{n})$ for all $n \in X$, and $T \vdash \neg\varphi(x)$ for all $n \in \mathbb{N} \smallsetminus X$.

A formula $\varphi(x, y)$ represents in T a partial function $f\colon \mathbb{N} \rightharpoonup \mathbb{N}$ if $T \vdash \forall y\,(\varphi(\overline{n}, y) \leftrightarrow y = \overline{m})$ for all $n, m \in \mathbb{N}$ such that $f(n) = m$.

42. All decidable sets are Σ_1-representable in Q.
[Hint: Starting with Σ_1-definitions of X and $\mathbb{N} \smallsetminus X$, write a Σ_1 formula expressing "there is a witness for $x \in X$ smaller than any witness for $x \notin X$". Use Exer. 32 (from 10 Dec) to show that it works.]

43. All partial computable functions are Σ_1-representable in Q.
[Hint: Using a Σ_1-definition of the graph of f, adapt the witness comparison argument from Exer. 42.]

44. Prove Gödel's diagonal lemma: for every formula $\varphi(x)$, there exists a sentence α such that $Q \vdash \alpha \leftrightarrow \varphi(\overline{\ulcorner\alpha\urcorner})$. [Hint: Using representability of a suitable computable function, construct a formula $\psi(x)$ such that $\mathsf{Q} \vdash \psi(\overline{\ulcorner\chi\urcorner}) \leftrightarrow \varphi(\overline{\ulcorner\chi(\overline{\ulcorner\chi\urcorner})\urcorner})$ for all $\chi(x)$.]