Exercises for Mathematical Logic (3 Oct 2025)

1. For every $\varphi \in \mathrm{Prop}_A$, its *De Morgan dual* $\varphi^{\mathrm{d}} \in \mathrm{Prop}_A$ is obtained by exchanging $\wedge$ with $\vee$ and $\top$ with $\bot$ inside φ. Formally, we define φ^{d} by induction on the complexity of φ:

$$
\begin{aligned}
a^{\mathrm{d}} &= a, \quad a \in A, & (\neg\varphi)^{\mathrm{d}} &= \neg(\varphi^{\mathrm{d}}),\\
\top^{\mathrm{d}} &= \bot, & \bot^{\mathrm{d}} &= \top,\\
(\varphi \wedge \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \vee \psi^{\mathrm{d}}), & (\varphi \vee \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \wedge \psi^{\mathrm{d}}).
\end{aligned}
$$

Show that for all assignments $\alpha\colon A \to \{0,1\}$, $\hat{\alpha}(\varphi^{\mathrm{d}}) = \hat{\alpha}_\neg(\neg\varphi)$, where $\alpha_\neg\colon A \to \{0,1\}$ is the assignment defined by $\alpha_\neg(a) = 1 - \alpha(a)$ for each $a \in A$.

2. Let $\varphi, \psi \in \mathrm{Prop}_A$.

(i) $\varphi \equiv \psi$ if and only if $\varphi^{\mathrm{d}} \equiv \psi^{\mathrm{d}}$.

(ii) $\varphi \vDash \psi$ if and only if $\psi^{\mathrm{d}} \vDash \varphi^{\mathrm{d}}$.