Exercises for Mathematical Logic (10 Oct 2025)

We have seen in the lecture that the De Morgan language $\{\wedge, \vee, \neg, \top, \bot\}$ is functionally complete, and specifically, that every Boolean function can be represented by a CNF or DNF of size $O(2^n n)$.

3. Prove that $\{\vee, \neg\}$, $\{\to, \bot\}$, and $\{\uparrow\}$ are functionally complete, where $x \uparrow y$ denotes the Sheffer stroke $\neg(x \wedge y)$.

4. Prove that $\{\to\}$, $\{\wedge, \vee, \top, \bot\}$, and $\{\leftrightarrow, \top, \bot\}$ are not functionally complete.
[Hint: Find a nontrivial property of Boolean functions which is preserved by composition, and holds for functions in the given basis.]

5. For any Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, the following are equivalent:

(i) $\{f\}$ is functionally complete.

(ii) $f(0,\dots,0) = 1$, $f(1,\dots,1) = 0$, and there exists an assignment α such that $\hat{\alpha}(f) = \hat{\alpha}_\neg(f)$ (where $\alpha_\neg$ is defined in Exer. 1).

[Hint: For (ii) $\to$ (i), look at functions obtained from f by identifying some of the variables.]

6. For any $n \in \mathbb{N}$, the *parity* function $\bigoplus_{i<n} x_i\colon \{0,1\}^n \to \{0,1\}$ is defined as $\left(\sum_{i<n} x_i\right) \bmod 2$. Show that any DNF or CNF representing $\bigoplus_{i<n} x_i$ has size $\Omega(2^n n)$. [Hint: What terms of the form $\bigwedge_{i \in I} x_i^{e_i}$ can imply one of $\bigoplus_{i<n} x_i = 0$ or $\bigoplus_{i<n} x_i = 1$? Here, $I \subseteq [n]$, $e_i \in \{0,1\}$, $x^1 = x$, $x^0 = \neg x$.]

7. There are formulas representing $\bigoplus_{i<n} x_i$ of size $O(n^c)$ for some constant c.
[Hint: Consider a balanced tree of binary parities. You may get it down to $c = 2$.]

8. Any DNF equivalent to the CNF $\bigwedge_{i<n}(x_i \vee y_i)$ has size $\Omega(2^n n)$.

9. Every Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ can be represented by a formula of size $O(2^n)$.
[Hint: Inductively express a formula in $n+1$ variables as a combination of formulas in n variables.]

10. Let $\Gamma, \Delta \subseteq \mathrm{Prop}_A$ and $\varphi, \psi \in \mathrm{Prop}_A$. Show that $\vdash$ satisfies Tarski's conditions for an abstract consequence relation:

(i) If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

(ii) If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.

(iii) If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for each $\psi \in \Gamma$, then $\Delta \vdash \varphi$.

11. Prove the propositional soundness theorem: for all $\Gamma \subseteq \mathrm{Prop}_A$ and $\varphi \in \mathrm{Prop}_A$, if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.