Exercises for Mathematical Logic (21 Nov 2025)

26. Prove that for every $k \geq 1$, the bijective base-k numeration is a bijection between $\mathbb{N}$ and $\{1, \dots, k\}^*$.

27. Show that the functions $+\colon \mathbb{N}^2 \to \mathbb{N}$ and $\cdot\colon \mathbb{N}^2 \to \mathbb{N}$ are computable when the input and output are represented in unary.

28. The same when the input and output are represented in binary.

29. Show that there are computable functions converting natural numbers from one representation to another (unary, ordinary base-k, bijective base-k, considering also different k's).

30. Fix an alphabet Σ.

(i) The following functions are computable: the constant function ε; the functions $s_a\colon \Sigma^* \to \Sigma^*$ for $a \in \Sigma$, defined by $s_a(x) = x\smile a$; the projections $\pi_i^n : (\Sigma^*)^n \to \Sigma^*$, $\pi_i^n(x_0, \dots, x_{n-1}) = x_i$.

(ii) If $f\colon (\Sigma^*)^n \to \Sigma^*$ and $g_i\colon (\Sigma^*)^m \to \Sigma^*$, $i < n$, are computable functions, their composition $h\colon (\Sigma^*)^m \to \Sigma^*$, $h(\vec{x}) = f(g_0(\vec{x}), \dots, g_{n-1}(\vec{x}))$, is computable.

(iii) If $f_\varepsilon\colon (\Sigma^*)^n \to \Sigma^*$ and $f_a\colon (\Sigma^*)^{n+2} \to \Sigma^*$, $a \in \Sigma$, are computable, the function $h\colon (\Sigma^*)^{n+1} \to \Sigma^*$ defined from them by the recursion

$$\begin{aligned} h(\vec{x}, \varepsilon) &= f_\varepsilon(\vec{x}), \\ h(\vec{x}, y\smile a) &= f_a(\vec{x}, y, h(\vec{x}, y)) \end{aligned}$$

is computable.

Functions in the smallest class that contains the functions from (i) and that is closed under the operations (ii) and (iii) are called *primitive recursive*. (Usually, the definition of primitive recursive functions is stated for functions $\mathbb{N}^n \to \mathbb{N}$, corresponding to our definition with $|\Sigma| = 1$ and the integers represented in unary. Our more general definition is equivalent up to the bijective base-$|\Sigma|$ numeration.)