Exercises for Mathematical Logic (28 Nov 2025)

31. Assume $|\Sigma| \geq 2$. Prove that the problem $H_\Sigma = \{\langle M\rangle x : M$ halts on input $x\} \subseteq \Sigma^*$ is many-one equivalent to A_Σ.

32. The set of well bracketed strings over the alphabet $\Sigma = \{(_i,)_i : i < k\}$ is the smallest set of strings such that the empty string ε is well bracketed, and if x and y are well bracketed and $i < k$, then xy and $(_i x)_i$ are well bracketed. E.g., $(_3(_1)_1(_2(_0)_0)_2(_1)_1)_3(_2)_2$ is well bracketed. Show that the set of well bracketed strings is decidable.

33. Let L be a finite first-order language. Show that the following sets and functions are computable:

(i) The set of L-terms.

(ii) The set of L-formulas.

(iii) The set of pairs $\langle \varphi, x\rangle$ where x is a free variable of an L-formula φ.

(iv) The substitution function: given an L-formula φ, a variable x, and an L-term t, compute $\varphi(t/x)$.

(v) The set of triples $\langle \Gamma, \varphi, \pi\rangle$ where π is a proof of an L-formula φ from a finite set of L-formulas Γ.

34. A language $X \subseteq \Sigma^*$ is semidecidable iff it can be represented as $\exists w \in \Sigma'^* \, P(x, w)$ for a finite alphabet Σ' (which we might take to be Σ itself if $|\Sigma| \geq 2$) and a decidable predicate P.
[Hint: Consider a description of an accepting run of a Turing machine, or—if you are already familiar with the section on arithmetic—a Σ_1-formula that defines X in $\mathbb{N}$.]

35. (Craig's trick.) Every semidecidable theory is recursively axiomatizable. [Hint: Express $\mathrm{Thm}(T)$ as $\exists w \, P(\varphi, w)$ with P decidable. Given φ and w, devise a sentence equivalent to φ that encodes w.]

36. Show that every decidable consistent theory T has a decidable complete extension.
[Hint: Consider a completion procedure that enumerates sentences φ one by one, and extends the current list of axioms with φ or $\neg\varphi$, whichever maintains consistency with T.]