Exercises for Mathematical Logic (5 Dec 2025)

37. Prove $\mathsf{Q} \vdash \forall x\,(x \leq \overline{n} \vee \overline{n} \leq x)$ for each $n \in \mathbb{N}$.

38. Q proves $x \cdot y = 0 \to x = 0 \vee y = 0$, and more generally, $x \cdot y = \overline{n} \to x = 0 \vee y \leq \overline{n}$ for each $n \in \mathbb{N}$.

39. The standard model $\mathbb{N}$ extends to an L_{PA}-structure $\mathbb{N}^\infty$ with domain $\mathbb{N} \cup \{\infty\}$, $\infty \notin \mathbb{N}$, so that $\mathbb{N}^\infty \vDash \mathsf{Q}$. Moreover, we are free to choose $(0 \cdot \infty)^{\mathbb{N}^\infty}$ in an arbitrary way (while the rest of the model is uniquely determined by the axioms of Q). Conclude that Q does not prove any of the formulas $S(x) \nleq x$, $x \cdot y = y \cdot x$, or $0 \cdot x \neq 1$.

40. Q does not prove $x + y = y + x$ or $0 + (x + y) = (0 + x) + y$.
[Hint: Modify the previous exercise to a model with two “infinities”.]