Exercises for Mathematical Logic (12 Dec 2025)

41. All Σ_1-definable sets $X \subseteq \mathbb{N}$ are semidecidable.

In the lecture, we developed an encoding of sequences in the language of arithmetic using Gödel's β-function. In the next three exercises, you will devise an alternative sequence encoding scheme due to E. Nelson, as simplified by P. Pudlák.

42. The set $\{x : \exists n \in \mathbb{N}\, x = 2^n\}$ of powers of 2 is definable by a Δ_0 formula, not using the 2^n function. [Hint: Consider the divisors of x.]

43. Consider an encoding of finite sets $X \subseteq \mathbb{N}$ by pairs $[r, w]$ where the binary expansion of w is a concatenation of binary expansions of elements of X, and the binary expansion of r acts as a "ruler" such that the positions of 1's mark where the individual elements of X start in w. Show that the predicate "x is in the set encoded by $[r, w]$" is Δ_0-definable.

44. Construct a Δ_0 encoding of finite sequences based on the previous exercise.

As yet another alternative, we will look at a representation of binary strings introduced by A. A. Markov Jr., who attributes it to J. Nielsen. The idea of using it for encoding strings in weak theories of arithmetic is due to J. Murwanashyaka; the extension to sequences of integers is due to A. Visser.

45. Let $\langle \mathrm{SL}_2(\mathbb{N}), I, \cdot \rangle$ denote the monoid of non-negative integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ of determinant 1, with $\cdot$ being matrix multiplication and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Put $A_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

- (i) Given $i = 0, 1$, which $M \in \mathrm{SL}_2(\mathbb{N})$ are of the form NA_i for $N \in \mathrm{SL}_2(\mathbb{N})$? [Hint: Focus on comparisons between the entries of M.]
- (ii) Using (i), show that each $M \in \mathrm{SL}_2(\mathbb{N}) \smallsetminus \{I\}$ can be written in a unique way as NA_0 or NA_1 with $N \in \mathrm{SL}_2(\mathbb{N})$.
- (iii) Conclude that $\mathrm{SL}_2(\mathbb{N}) \simeq \langle \{0,1\}^*, \varepsilon, \smile \rangle$.

46. Develop a Δ_0 encoding of finite sequences based on the previous exercise. [Hint: You may represent $\{n_0, \dots, n_{k-1}\}$ by $A_0^{n_0} \cdots A_1 A_0^{n_{k-1}} A_1$, using $A_0^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. Then encode sequences by sets.]