Exercises for Mathematical Logic (19 Dec 2025)

A formula $\varphi(x)$ *represents* a set $X \subseteq \mathbb{N}$ in a theory T if $T \vdash \varphi(\overline{n})$ for all $n \in X$, and $T \vdash \neg\varphi(\overline{n})$ for all $n \in \mathbb{N} \smallsetminus X$.

A formula $\varphi(x, y)$ represents in T a partial function $f\colon \mathbb{N} \rightharpoonup \mathbb{N}$ if $T \vdash \forall y\, (\varphi(\overline{n}, y) \leftrightarrow y = \overline{m})$ for all $n, m \in \mathbb{N}$ such that $f(n) = m$.

47. All decidable sets are Σ_1-representable in Q.
[Hint: Starting with Σ_1 definitions of X and $\mathbb{N} \smallsetminus X$, write a Σ_1 formula expressing "there is a witness for $x \in X$ smaller than any witness for $x \notin X$". Use Exer. 37 (from 5 Dec) to show that it works.]

48. All partial computable functions are Σ_1-representable in Q.
[Hint: Using a Σ_1 definition of the graph of f, adapt the witness comparison argument from Exer. 47.]