Exercises for Mathematical Logic (9 Jan 2026)

49. Prove Gödel's diagonal lemma: for every formula $\varphi(x)$, there exists a sentence α such that $Q \vdash \alpha \leftrightarrow \varphi(\overline{\ulcorner \alpha \urcorner})$. [Hint: Using representability of a suitable computable function (see Exer. 48), construct a formula $\psi(x)$ such that $Q \vdash \psi(\overline{\ulcorner \chi \urcorner}) \leftrightarrow \varphi(\overline{\ulcorner \chi(\overline{\ulcorner \chi \urcorner}) \urcorner})$ for all $\chi(x)$.]

50. (Löb's theorem.) Let T be an extension of Q, and Pr_T a provability predicate for T that satisfies the Hilbert–Bernays–Löb derivability conditions. Then for any sentence φ, if $T \vdash \mathrm{Pr}_T(\overline{\ulcorner \varphi \urcorner}) \to \varphi$, then $T \vdash \varphi$. [Hint: Generalize the proof of Gödel's second incompleteness theorem. Alternatively, show that $\mathrm{Pr}_T(\ulcorner(\neg\varphi \to \dot{x})\urcorner)$ is a Hilbert–Bernays–Löb proof predicate for $T + \neg\varphi$, and apply the second incompleteness theorem directly.]

51. (Tarski's theorem on undefinability of truth.) Let T be a consistent recursively axiomatizable extension of Q. Then there is no formula $\mathrm{Tr}(x)$ such that $T \vdash \mathrm{Tr}(\overline{\ulcorner \varphi \urcorner}) \leftrightarrow \varphi$ for all sentences φ.
[Hint: Construct a liar sentence.]