Grothendieck spaces, operators, and beyond

Tomasz Kania

Academy of Sciences of the Czech Republic, Praha

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joint work with K. Beanland & N. J. Laustsen

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  So is it one of those 'non-separable' talks? Well...

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$C(K)$ for K Stonean, σ-Stonean, F-space, all indecomposable spaces $C(K)$ constructed by Koszmider, Plebanek, and others.

No topological characterisation of when a $C(K)$-space is Grothendieck exists.

If $C(K)$ is Grothendieck, K does not have non-trivial convergent sequences.

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Dual spaces with Pełczyński's property (V). (X has prop. (V) whenever for any space Y every unconditionally converging operator $T\colon X \to Y$ is weakly compact.)

- $C(K)$-spaces have prop. (V).
- C*-algebras have prop. (V) (Pfitzner, 1994);
  all von Neumann algebras are Grothendieck spaces
  $\mathscr{B}(H)$, the space of op$^{\mathrm{s}}$ on a Hilbert space H, is Grothendieck.

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> *Finally, there is some evidence (Akemann [1967], [1968]) that the space $\mathscr{L}(H; H)$ of bounded linear operators on a Hilbert space is a Grothendieck space and that more generally the space $\mathscr{L}(X; X)$ is a Grothendieck space for any reflexive Banach space.*

Evidence was not good enough

Theorem (K. 2012). *Let* $p \in (1, \infty)$. *Set* $E_p = \big(\bigoplus_n \ell_1^n\big)_{\ell_p}$. *Then the space of operators* $\mathscr{B}(E_p)$ *is not Grothendieck, even though* E_p *is reflexive.*

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Question: Is $\mathscr{B}(X)$ ever reflexive? (X ∞-dim)
see M. Ostrovskii's post `mathoverflow.net/q/232291/15129`

Maybe this idea is not bad after all

Theorem (Beanland, K., Laustsen). *Let X be the Tsirelson space or the p^{th} Baernstein space ($p \in (1, \infty)$). Then $\mathscr{B}(X)$ is not a Grothendieck space*

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- E a space with a normalised, 1-unconditional basis $(e_n)_{n=1}^{\infty}$.
  The *E-direct sum* of a sequence $(X_n)_{n=1}^{\infty}$ of spaces is given by

$$\Big(\bigoplus_{n\in\mathbb{N}} X_n\Big)_E = \Big\{(x_n)\colon x_n \in X_n \ (n \in \mathbb{N}) \text{ and } \sum_{n=1}^{\infty} \|x_n\| e_n \text{ converges in } E\Big\}.$$

  This is a Banach space w.r.t $\|(x_n)\| = \big\| \sum_{n=1}^{\infty} \|x_n\| e_n \big\|$.

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Abstraction. Let $X = \big(\bigoplus_{n\in\mathbb{N}} X_n\big)_E$, where E is a Banach space with a normalized, 1-unconditional basis and (X_n) is a sequence of Banach spaces. Then $\mathscr{B}(X)$ contains a complemented subspace which is isometrically isomorphic to $\big(\bigoplus_{n\in\mathbb{N}} X_n\big)_{\ell_\infty}$.

Key observation. We observe that the (Johnson–Figiel) Tsirelson space X is 54-isomorphic to an unconditional FDD $(\bigoplus F_n)_T$, where E_n is 2-isomorphic to $\ell_1(m_n)$ and $m_n \to \infty$ as $n \to \infty$.

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***Baernstein spaces**. For $x \in c_{00}$ and $N_j \subset \mathbb{N}$ set*

$$\nu_p(x; N_1, \ldots, N_k) = \left(\sum_{j=1}^{k} \mu(x, N_j)^p \right)^{\frac{1}{p}}, \qquad \textit{where} \qquad \mu(x, N_j) = \sum_{n \in N_j} |\alpha_n|.$$

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Definition. B_p is the completion of c_{00} w.r.t.

$$\|x\|_{B_p} = \sup\{\nu_p(x; N_1, \ldots, N_k) \colon k \in \mathbb{N} \text{ and } N_1 < N_2 < \cdots < N_k, |N_j| < \min N_j\}$$

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a word about the proof and some perspectives

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Caveat. B_p is not isomorphic to any E-sum of f.d. blocks of the basis isomorphic to $\ell_1^{m_n}$ $(m_n \to \infty)$ for any space E with a normalised 1-unconditional basis.

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1. $\mathscr{B}(X_{\mathrm{AH}}) = \mathbb{C} \cdot I_{X_{\mathrm{AH}}} \oplus \mathscr{K}(X_{\mathrm{AH}})$
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A distantly related question. Let $E_n = C^1([0,1]^n)$. Is E_n^{**} Grothendieck? (Yes, for $n = 1$.) If this is the case, this would have solved a recent question by Candido, Cúth, and Doucha of whether $\mathrm{Lip}\,\mathbb{Z}^n$ is Grothendieck.

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Question. Is $\mathscr{B}(X)$ Grothendieck for super-reflexive X?

Embarrassingly, we do not know whether $\mathscr{B}(\ell_p)$ is Grothendieck for $p \in (1,2) \cup (2,\infty)$... (btw. $\mathscr{B}(\ell_p) \cong \mathscr{B}(L_p)$ as Banach spaces).

A distantly related question. Let $E_n = C^1([0,1]^n)$. Is E_n^{**} Grothendieck? (Yes, for $n=1$.) If this is the case, this would have solved a recent question by Candido, Cúth, and Doucha of whether $\mathrm{Lip}\,\mathbb{Z}^n$ is Grothendieck.

Very few operators vs. Grothendieck

Argyros and Haydon built a space X_{AH} such that

1. $\mathscr{B}(X_{\mathrm{AH}}) = \mathbb{C} \cdot I_{X_{\mathrm{AH}}} \oplus \mathscr{K}(X_{\mathrm{AH}})$
2. X_{AH} has a basis,
3. $X_{\mathrm{AH}}^* \cong \ell_1$.

Ultimate (?) problem in the area: is there a reflexive space with 1. & 2.?

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Thank you very much!