Isomorphic Kottman constant of a Banach space

Tomasz Kania

Matematický ústav AV ČR, Praha

48. Zimní škola abstraktní analýzy, Svratka, 12. ledna 2020

joint work with J.M.F. Castillo, M. Gonzalez, and P. Papini

(Pre-)history

X stands for an **inf.-dim.** Banach space; S_X the unit sphere of X.

(Pre-)history

X stands for an **inf.-dim.** Banach space; S_X the unit sphere of X.

Riesz' lemma (1916). For every $\theta \in (0,1)$ there exists $(x_n)_{n=1}^{\infty} \subset S_X$ with

$$\|x_n - x_k\| \geqslant \theta \qquad (n \neq k).$$

(Pre-)history

X stands for an **inf.-dim.** Banach space; S_X the unit sphere of X.

Riesz' lemma (1916). For every $\theta \in (0,1)$ there exists $(x_n)_{n=1}^\infty \subset S_X$ with

$$\|x_n - x_k\| \geqslant \theta \qquad (n \neq k).$$

In other words: S_X contains a θ-*separated* sequence.

(Pre-)history

X stands for an **inf.-dim.** Banach space; S_X the unit sphere of X.

Riesz' lemma (1916). For every $\theta \in (0,1)$ there exists $(x_n)_{n=1}^\infty \subset S_X$ with

$$\|x_n - x_k\| \geqslant \theta \qquad (n \neq k).$$

In other words: S_X contains a θ-*separated* sequence.

Kottman's theorem (1975). There exists $(x_n)_{n=1}^\infty \subset S_X$ such that

$$\|x_n - x_k\| > 1 \qquad (n \neq k).$$

(Pre-)history

X stands for an **inf.-dim.** Banach space; S_X the unit sphere of X.

Riesz' lemma (1916). For every $\theta \in (0,1)$ there exists $(x_n)_{n=1}^{\infty} \subset S_X$ with

$$\|x_n - x_k\| \geqslant \theta \qquad (n \neq k).$$

In other words: S_X contains a θ-*separated* sequence.

Kottman's theorem (1975). There exists $(x_n)_{n=1}^{\infty} \subset S_X$ such that

$$\|x_n - x_k\| > 1 \qquad (n \neq k).$$

Let us call this situation: S_X contains an infinite $(1+)$-*separated* subset.

(Pre-)history

X stands for an **inf.-dim.** Banach space; S_X the unit sphere of X.

Riesz' lemma (1916). For every $\theta \in (0,1)$ there exists $(x_n)_{n=1}^{\infty} \subset S_X$ with

$$\|x_n - x_k\| \geqslant \theta \qquad (n \neq k).$$

In other words: S_X contains a θ-*separated* sequence.

Kottman's theorem (1975). There exists $(x_n)_{n=1}^{\infty} \subset S_X$ such that

$$\|x_n - x_k\| > 1 \qquad (n \neq k).$$

Let us call this situation: S_X contains an infinite $(1+)$-*separated* subset.

The Elton–Odell theorem (1981). There exists $\varepsilon = \varepsilon(X)$ such that S_X contains a $(1+\varepsilon)$-separated sequence.

Symmetric separation in the separable case

What about symmetric separation? ($\|x \pm y\| > \delta$)?

Theorem (Hájek–K.–Russo, '18)

- *The unit sphere of an infinite-dimensional space contains a symmetrically $(1+)$-separated sequence.*

Symmetric separation in the separable case

What about symmetric separation? ($\|x \pm y\| > \delta$)?

Theorem (Hájek–K.–Russo, '18)

- *The unit sphere of an infinite-dimensional space contains a symmetrically (1+)-separated sequence.*
- *The unit sphere of an infinite-dimensional Banach* ***lattice*** *or* ***separable dual space*** *contains a symmetrically $(1+\varepsilon)$-separated sequence.*

Symmetric separation in the separable case

What about symmetric separation? ($\|x \pm y\| > \delta$)?

Theorem (Hájek–K.–Russo, '18)

- *The unit sphere of an infinite-dimensional space contains a symmetrically $(1+)$-separated sequence.*
- *The unit sphere of an infinite-dimensional Banach* ***lattice*** *or* ***separable dual space*** *contains a symmetrically $(1+\varepsilon)$-separated sequence.*
- *If X has finite cotype $q(X)$ (e.g., X is super-reflexive), then S_X contains a $(2^{1/q(X)}-)$-separated sequence.*

Symmetric separation in the separable case

What about symmetric separation? ($\|x \pm y\| > \delta$)?

Theorem (Hájek–K.–Russo, '18)

- *The unit sphere of an infinite-dimensional space contains a symmetrically $(1+)$-separated sequence.*
- *The unit sphere of an infinite-dimensional Banach* ***lattice*** *or* ***separable dual space*** *contains a symmetrically $(1+\varepsilon)$-separated sequence.*
- *If X has finite cotype $q(X)$ (e.g., X is super-reflexive), then S_X contains a $(2^{1/q(X)}-)$-separated sequence.*

cotype = inf of q s.t. $\mathbb{E}\|\sum_{i\in F} r_i x_i\|^2 \leqslant q^{-2}\sum_{i\in F}\|x_i\|^2$ for all finite tuples $(x_i)_{i\in F}$ in X, where (r_i) is a sequence of i.i.d. symmetric Bernoulli random variables.

Symmetric separation in the separable case

What about symmetric separation? ($\|x \pm y\| > \delta$)?

Theorem (Hájek–K.–Russo, '18)

- *The unit sphere of an infinite-dimensional space contains a symmetrically $(1+)$-separated sequence.*
- *The unit sphere of an infinite-dimensional Banach **lattice** or **separable dual space** contains a symmetrically $(1+\varepsilon)$-separated sequence.*
- *If X has finite cotype $q(X)$ (e.g., X is super-reflexive), then S_X contains a $(2^{1/q(X)}-)$-separated sequence.*

cotype = inf of q s.t. $\mathbb{E}\|\sum_{i\in F} r_i x_i\|^2 \leqslant q^{-2}\sum_{i\in F}\|x_i\|^2$ for all finite tuples $(x_i)_{i\in F}$ in X, where (r_i) is a sequence of i.i.d. symmetric Bernoulli random variables.

Theorem (Russo, '19)

The unit sphere of an infinite-dimensional Banach **space** contains a symmetrically $(1+\varepsilon)$-separated sequence.

Separation under renormings

Given X, one may find an equiv. norm $\tilde{X}$ so that $S_{\tilde{X}}$ contains a 2-sep. sequence.

Separation under renormings

Given X, one may find an equiv. norm $\tilde{X}$ so that $S_{\tilde{X}}$ contains a 2-sep. sequence.

Take a b.-o. seq. $(x_n, f_n) \in X \times X^*$ ($\langle f_k, x_j\rangle = \delta_{kj}$, $\|f_i\| = \|x_i\| = 1$) and set

$\nu(X) = \sup_{i\neq k} |\langle f_i, x\rangle| + |\langle f_k, x\rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

- $K(X) = \sup\{\sigma > 0\colon S_X$ contains a σ-separated sequence$\}$ (Kottman constant)

Separation under renormings

Given X, one may find an equiv. norm $\tilde{X}$ so that $S_{\tilde{X}}$ contains a 2-sep. sequence.

Take a b.-o. seq. $(x_n, f_n) \in X \times X^*$ ($\langle f_k, x_j\rangle = \delta_{kj}$, $\|f_i\| = \|x_i\| = 1$) and set

$\nu(X) = \sup_{i \neq k} |\langle f_i, x\rangle| + |\langle f_k, x\rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

- $K(X) = \sup\{\sigma > 0\colon S_X \text{ contains a } \sigma\text{-separated sequence}\}$ (Kottman constant)
- $\tilde{K}(X) = \inf\{K(\tilde{X})\colon X \cong \tilde{X}\}$ (isomorphic Kottman constant).

Separation under renormings

Given X, one may find an equiv. norm $\tilde{X}$ so that $S_{\tilde{X}}$ contains a 2-sep. sequence.

Take a b.-o. seq. $(x_n, f_n) \in X \times X^*$ ($\langle f_k, x_j\rangle = \delta_{kj}$, $\|f_i\| = \|x_i\| = 1$) and set

$\nu(X) = \sup_{i\neq k} |\langle f_i, x\rangle| + |\langle f_k, x\rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

- $K(X) = \sup\{\sigma > 0\colon S_X$ contains a σ-separated sequence$\}$ (Kottman constant)
- $\tilde{K}(X) = \inf\{K(\tilde{X})\colon X \cong \tilde{X}\}$ (isomorphic Kottman constant).
- $K_f(X) = \sup\{\sigma > 0\colon \forall N \in \mathbb{N}\ \exists (x_n)_{n=1}^N$ in B_X s.t. $\|x_n - x_m\| \geqslant \sigma$ for $n \neq m\}$ (finite Kottman constant)
- $K_s, \tilde{K}_s$ the symmetric versions.

- $K(c_0) = 2 = \tilde{K}_s(c_0)$,

Separation under renormings

Given X, one may find an equiv. norm $\tilde{X}$ so that $S_{\tilde{X}}$ contains a 2-sep. sequence.

Take a b.-o. seq. $(x_n, f_n) \in X \times X^*$ $(\langle f_k, x_j \rangle = \delta_{kj}, \|f_i\| = \|x_i\| = 1)$ and set

$\nu(X) = \sup_{i \neq k} |\langle f_i, x \rangle| + |\langle f_k, x \rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

- $K(X) = \sup\{\sigma > 0\colon S_X$ contains a σ-separated sequence$\}$ (Kottman constant)
- $\tilde{K}(X) = \inf\{K(\tilde{X})\colon X \cong \tilde{X}\}$ (isomorphic Kottman constant).
- $K_f(X) = \sup\{\sigma > 0\colon \forall N \in \mathbb{N}\ \exists (x_n)_{n=1}^N$ in B_X s.t. $\|x_n - x_m\| \geqslant \sigma$ for $n \neq m\}$ (finite Kottman constant)
- K_s, $\tilde{K}_s$ the symmetric versions.

- $K(c_0) = 2 = \tilde{K}_s(c_0)$,
- $K(\ell_p) = 2^{1/p} = \tilde{K}_s(\ell_p)$;

Separation under renormings

Given X, one may find an equiv. norm $\tilde{X}$ so that $S_{\tilde{X}}$ contains a 2-sep. sequence.

Take a b.-o. seq. $(x_n, f_n) \in X \times X^*$ ($\langle f_k, x_j\rangle = \delta_{kj}$, $\|f_i\| = \|x_i\| = 1$) and set

$\nu(X) = \sup_{i\neq k} |\langle f_i, x\rangle| + |\langle f_k, x\rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

- $K(X) = \sup\{\sigma > 0 \colon S_X$ contains a σ-separated sequence$\}$ (Kottman constant)
- $\tilde{K}(X) = \inf\{K(\tilde{X}) \colon X \cong \tilde{X}\}$ (isomorphic Kottman constant).
- $K_f(X) = \sup\{\sigma > 0 \colon \forall N \in \mathbb{N}\ \exists (x_n)_{n=1}^N$ in B_X s.t. $\|x_n - x_m\| \geqslant \sigma$ for $n \neq m\}$ (finite Kottman constant)
- K_s, $\tilde{K}_s$ the symmetric versions.

- $K(c_0) = 2 = \tilde{K}_s(c_0)$,
- $K(\ell_p) = 2^{1/p} = \tilde{K}_s(\ell_p)$;
- Kryczka–Prus: $K(X) \geqslant \sqrt[5]{4}$ for any non-reflexive X.

Preliminary observations

- For a countably incomplete ultrafilter $\mathscr{U}$ and a space X, we have

$$1 < K(X) \leqslant K_f(X) = K(X^{\mathscr{U}}) \leqslant 2,$$

where $X^{\mathscr{U}}$ stands for the ultrapower of X w.r.t. $\mathscr{U}$.

Preliminary observations

- For a countably incomplete ultrafilter $\mathscr{U}$ and a space X, we have

$$1 < K(X) \leqslant K_f(X) = K(X^{\mathscr{U}}) \leqslant 2,$$

  where $X^{\mathscr{U}}$ stands for the ultrapower of X w.r.t. $\mathscr{U}$.
- There exists a space Z for which

$$K(Z) < K(Z^{**}),$$

  and it is easy to check that this space also satisfies $K_s(Z) < K_s(Z^{**})$. The said space is a J-sum of ℓ_1^n ($n \in \mathbb{N}$) in the sense of Bellenot; it has the property that $K(Z) < 2$, yet Z^{**} admits a quotient map onto ℓ_1 so that $K_s(Z^{**}) = 2$.

Castillo–González–K.–Papini

For every space X, $2 \leqslant K(X) \cdot K(X^*)$.

Based on a simple application of Ramsey's theorem:

Lemma

Let (x_n) be a bounded sequence in a Banach space. Then there exists an infinite subset M of $\mathbb{N}$ such that $\|x_i - x_j\|$ converges as $i, j \in M$, $i, j \to \infty$.

Proof.

X contains a basic seq. with basis constant at most $1 + \varepsilon$: $(x_n)_{n=1}^{\infty}$ in X and $(x_n^*)_{n=1}^{\infty}$ in X^* with $\|x_n\| = 1$ and $\|x_n^*\| \leqslant 1 + \varepsilon$ ($n \in \mathbb{N}$) s.t. $\langle x_i^*, x_j \rangle = \delta_{ij}$. For $i \neq j$,

$$2 = \langle x_i^* - x_j^*, x_i - x_j \rangle \leqslant \|x_i^* - x_j^*\| \cdot \|x_i - x_j\|.$$

Let us set $y_n^* = (1 + \varepsilon)^{-1} x_n^*$. (Passing to a subsequence) $\|y_i^* - y_j^*\|$ and $\|x_i - x_j\|$ converge to k^* and to k, resp. in the sense of the Lemma. Then

$$2(1 + \varepsilon)^{-1} \leqslant k^* \cdot k \leqslant K(X^*) \cdot K(X),$$

hence $2 \leqslant K(X) \cdot K(X^*)$.

Twisted sums

Castillo–González–K.–Papini

For a short exact sequence of Banach spaces

$$0 \to Y \to X \to Z \to 0,$$

we have

$$\tilde{K}(X) = \max\{\tilde{K}(Y), \tilde{K}(Z)\}.$$

Main idea: the constant is cts w.r.t. to the Kadets metric

$$d_K(M, N) = \inf \max \Big\{ \sup_{x \in iB_M} \operatorname{dist}(x, jB_N), \sup_{y \in jB_N} \operatorname{dist}(y, iB_M) \Big\},$$

where the inf is taken w.r.t all isometric embeddings i, j of M, N into common spaces.

- J.M.F. Castillo and P.L. Papini, On the Kottman constants in Banach spaces, *Banach Center Publ*. 92 (2011), 75–84.
- P. Hájek, T. Kania, and T. Russo, Symmetrically separated sequences in the unit sphere of a Banach space, *J. Funct. Anal.* 275 (2018), 3148–3168.
- C. A. Kottman, Subsets of the unit ball that are separated by more than one. *Studia Math.* 53 (1975), 15–27.

Preprints:

- P. Hájek, T. Kania, and T. Russo, Separated sets and Auerbach systems in Banach spaces, `arXiv:1803.11501`.
- J. M. F. Castillo, M. González, T. Kania, and P. Papini, The isomorphic Kottman constant of a Banach space, `arXiv:1910.01626`.