(1+)-separation in the unit sphere of a Banach space

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(joint work with P. Hájek and T. Russo)

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The Elton–Odell theorem (1981). There exists $\varepsilon = \varepsilon(X)$ such that S_X contains a $(1+\varepsilon)$-separated sequence.

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Conclusion: The most optimistic variant of the question was perhaps *too optimistic*.

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- *Suppose that X is 'large enough' ($w^*\text{-}\mathrm{dens}\, X^* > \exp_2 \mathfrak{c}$). Then both S_X and S_{X^*} contain an uncountable $(1+)$-separated family.*

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Example (K.–Kochanek): The space

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Theorem (Hájek)

There exists a reflexive space X with $\operatorname{dens} X = \omega_1$ such that for every $p \in (1,\infty)$ there is no injective linear operator $X \to \ell_p(\omega_1)$.

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We use this to construct a linear injection from a subspace of X that maps some unit vectors onto the unit vector basis of $\ell_p(\lambda)$.

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Thank you for your attention!